What are Multivariate Splines?

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Splines are smooth piecewise polynomial functions.

Disclaimers, Sources, Acknowledgments

- I am going to ignore spline alternatives, such as tensor product schemes, radial basis functions, parametric interpolation and approximation schemes, and alternative spline definitions.
- For a lack of time, I am also going to skip over the rich history of the subject.
- A comprehensive and current source of information on splines is the new monograph: Ming-Jun Lai and Larry L. Schumaker, Spline Functions on Triangulations, Cambridge University Press, 2007, ISBN 9780521875929.
- The software demonstrated in this talk can be used and downloaded at

  www.math.utah.edu/~pa

- Much of my spline research since 1983 has been in collaboration with Larry Schumaker.
- Recently I have had the pleasure to work on several projects with Tanya Sorokina.
- *Two Tetrahedral* C^1 *Cubic Macro Elements*, submitted

Outline of Talk

- Splines
- Dimension Problem for Univariate Splines
- Bivariate Splines
- The Bernstein Bézier Form
- Smoothness Conditions
- Dimension Problem for Bivariate Splines
- S^1_3
- Trivariate Splines
- Macro Elements
- Open Problems

Univariate Splines

$$a = x_0 < x_1 < \ldots < x_N = b$$

- partitions $[a,b]$ into N subintervals

$$I_i = [x_{i-1}, x_i], \qquad i = 1, \ldots, N$$

- spline space:

$$S_d^r = \{s \in C^r[a,b] : s|_{I_i} \in P_d, \quad i = 1, \ldots, N\}$$

where P_d is the space of univariate polynomials of degree d.

- Obviously:

$$\begin{aligned}\dim S_d^r &= (d+1) + (N-1)(d+1-(r+1)) \\ &= N(d-r) + r + 1.\end{aligned}$$

What could be simpler?

Examples:

- **piecewise linear:** $\dim S_1^0 = N + 1$,
- **cubic splines:** $\dim S_3^2 = N + 2$,
- **piecewise cubic Hermite:** $\dim S_3^1 = 2(N + 1)$

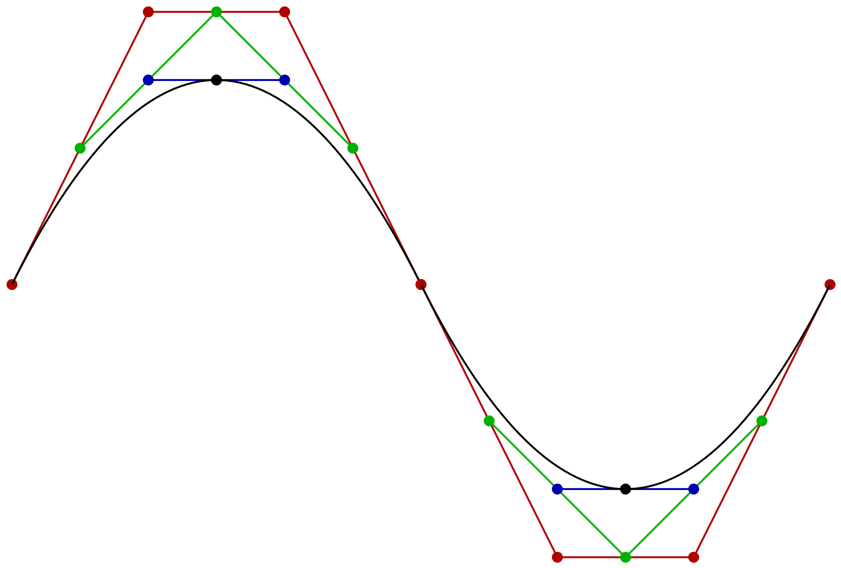

Figure 1. A univariate spline in S_3^1.

- What about two variables?
- Need to partition a polygonal domain
- rectangular grid leads to tensor product schemes. They form a huge area. But they are essentially univariate splines done twice.
- If points are distributed irregularly then **triangles** work.

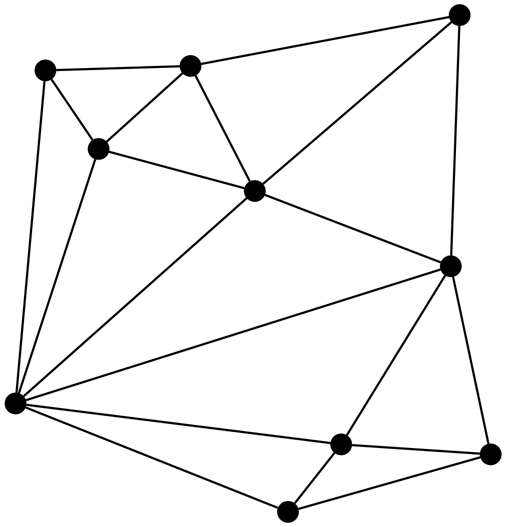

Figure 2. A triangulation.

Let Ω be the union of the triangles T_i, $i = 1, \ldots, N$, and P^d the $\binom{d+2}{2}$-dimensional space of bivariate polynomials of degree d.

$$P_d = \left\{ p : p(x_1, x_2) = \sum_{i=0}^{d} \sum_{j=0}^{d-i} \alpha_{ij} x_1^i x_2^j \right\}.$$

$$S_d^r = \left\{ s \in C^r(\Omega) : s|_{T_i} \in P^d \right\}.$$

The Bernstein-Bézier Form

Let Δ be a triangle with vertices v_1, v_2, and v_3.
For $x \in \mathbb{R}^2$, define its *barycentric coordinates*:

$$x = \sum_{i=1}^{3} b_i v_i \quad \text{where} \quad \sum_{i=1}^{3} b_i = 1.$$

Barycentric coordinates are linear functions of x.

Any polynomial $p \in P_d$ can be written uniquely in its *Bernstein-Bézier form* as:

$$p(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k.$$

where

c_{ijk} : *Bézier ordinates*

$P_{ijk} = \dfrac{iv_1 + jv_2 + kv_3}{d} \in \mathbb{R}^2$: *Domain Points*

$(P_{ijk}, c_{ijk}) \in \mathbb{R}^3$: *Bézier control points*

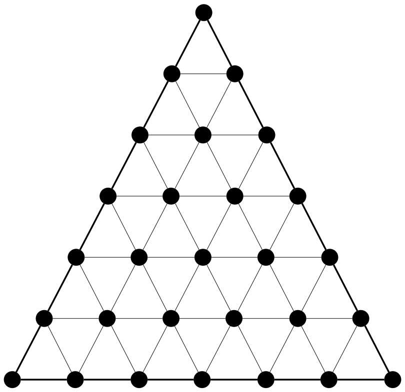

Figure 3. Domain points, $d = 6$.

Simple Geometry

- The control points at the vertices lie on the graph of the polynomial. This is because

$$p(v_1) = c_{d00}, \quad p(v_2) = c_{0d0}, \quad p(v_3) = c_{0d0}$$

- The control points in the 1-disk, i.e.,

$$(P_{d00}, c_{d00}), \quad (P_{d-1,1,0}, c_{d-1,1,0}), \quad (P_{d-1,0,1}, c_{d-1,0,1})$$

  lie in the tangent plane of p at v_1. Similarly for v_2 and v_3.

- The control points along an edge determine the values of the polynomial along that edge.
- The control points along an edge, and in the first row parallel to the edge, determine the values of first derivatives of the polynomial along that edge.

Smoothness Conditions

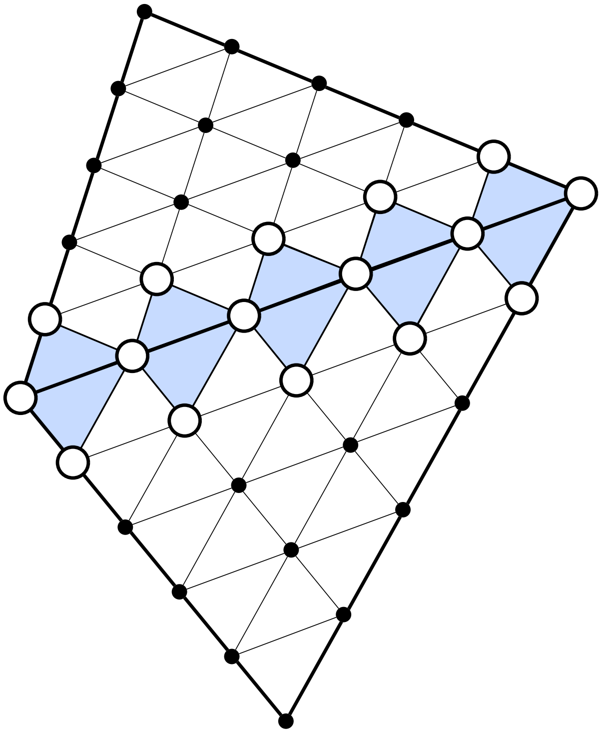

Figure 4. Smoothness conditions.

C^0: Control points along edges coincide

C^1: Quadrilaterals across edges are planar in $\mathbb{R}^3$.

C^r: Evaluate subpolynomials.

Idea of a particular proof:

Let D be a first order derivative operator.

$$p(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k, \qquad \text{as before}$$

On a neighboring triangle have

$$\tilde{p}(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} \tilde{c}_{ijk} b_1^i b_2^j b_4^k.$$

For continuity, $c_{ij0} = \tilde{c}_{ij0}$, $i + j = d$.

$$\begin{aligned} Dp(x) = &\sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} \\ &\left(i b_1^{i-1} D b_1 b_2^j b_3^k + b_1^i j b_2^{j-1} D b_2 b_3^k + b_1^i b_2^j k b_3^{k-1} D b_k \right) \\ = &\sum_{i+j+k=d-1} \hat{c}_{ijk} b_1^i b_2^j b_3^k \end{aligned}$$

where

$$\hat{c}_{ijk} = d(c_{i+1,j,k} Db_1 + c_{i,j+1,k} Db_2 + c_{i,j,k+1} Db_3)$$

Differentiating on both triangles, restricting the derivative to the common edge, and equating coefficients gives the condition

$$d(a_{i+1j0} c_{i+1,j0} + a_{ij+10} c_{ij+10} + a_{ij11} c_{ij1} + \tilde{a}_{ij1} \tilde{c}_{ij1}) = 0 \qquad (*)$$

for $i + j = d - 1$.

The coefficients in $(*)$ are independent of the particular quadrilateral and the polynomial degree.

In particular, we obtain the same condition for the case $d = 1$. In that case, the condition means that a piecewise linear function is linear.

This means that in the large quadrilateral, formed by both triangles, for $d = 1$, the four control points lie in the same plane.

Since the small quadrilaterals are similar to the large one, the condition $(*)$ has the same geometric meaning.

Note the interplay of Geometry and Algebra!

Major Difficulty: The dimension of S_d^r depends not just on the **combinatorics** of a triangulation, but also on its **geometry**.

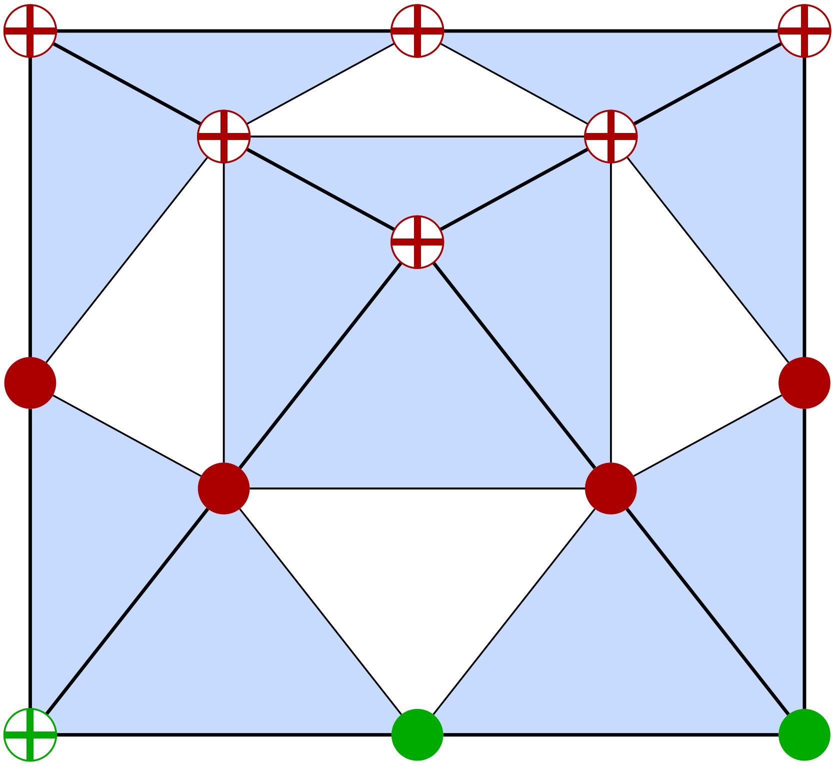

Figure 5. $\dim S_2^1 = 7$ on a generic 4-star.

download software: www.math.utah.edu/~pa/MDS

A **singular vertex** is an interior vertex of degree 4 whose edges form two intersecting lines.

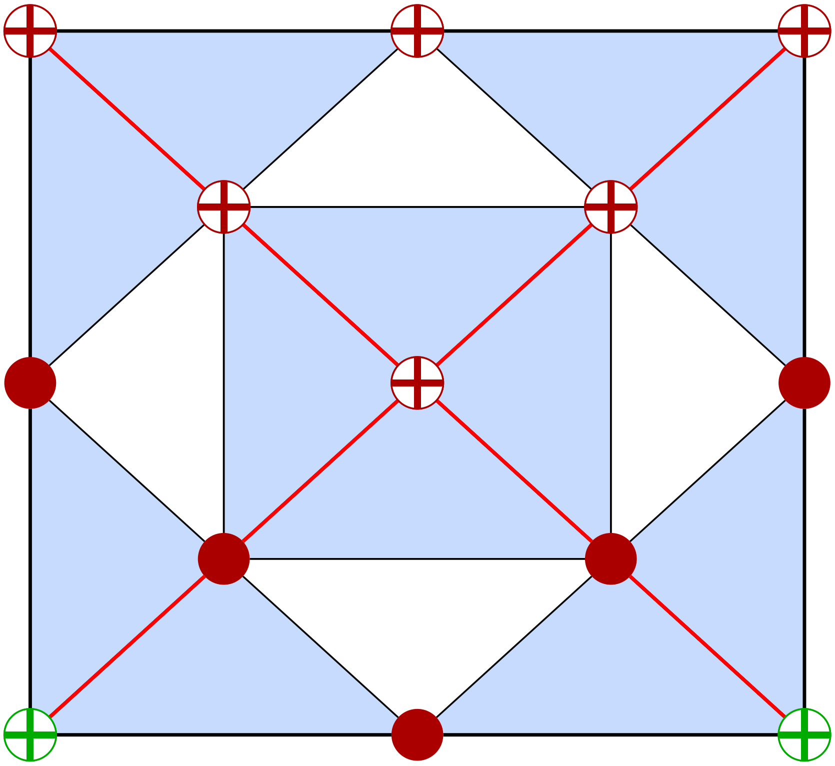

Figure 6. $\dim S^1_2 = 8$ on a singular vertex.

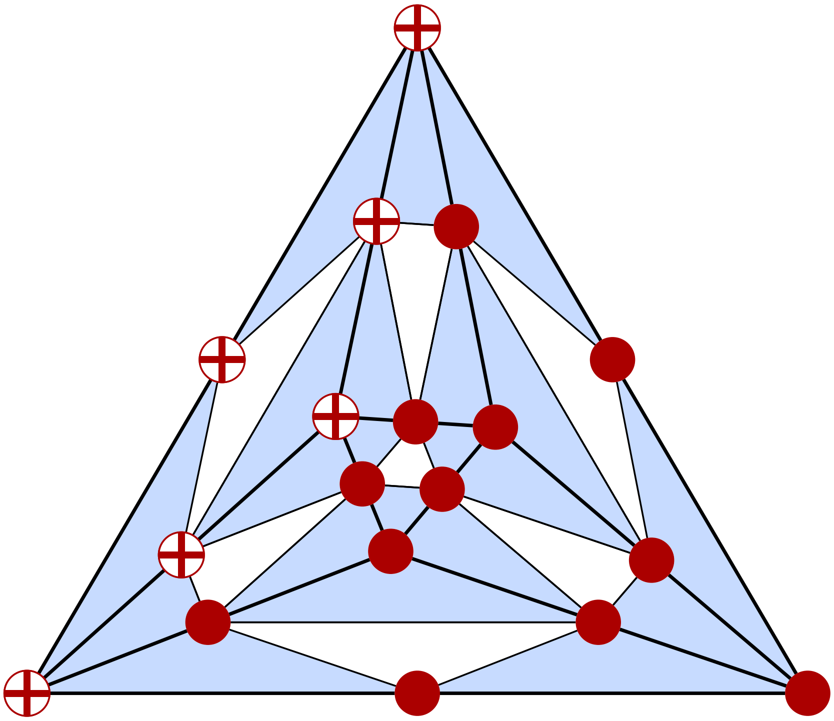

Figure 7. $\dim S_2^1 = 6$ on the generic Morgan-Scott split.

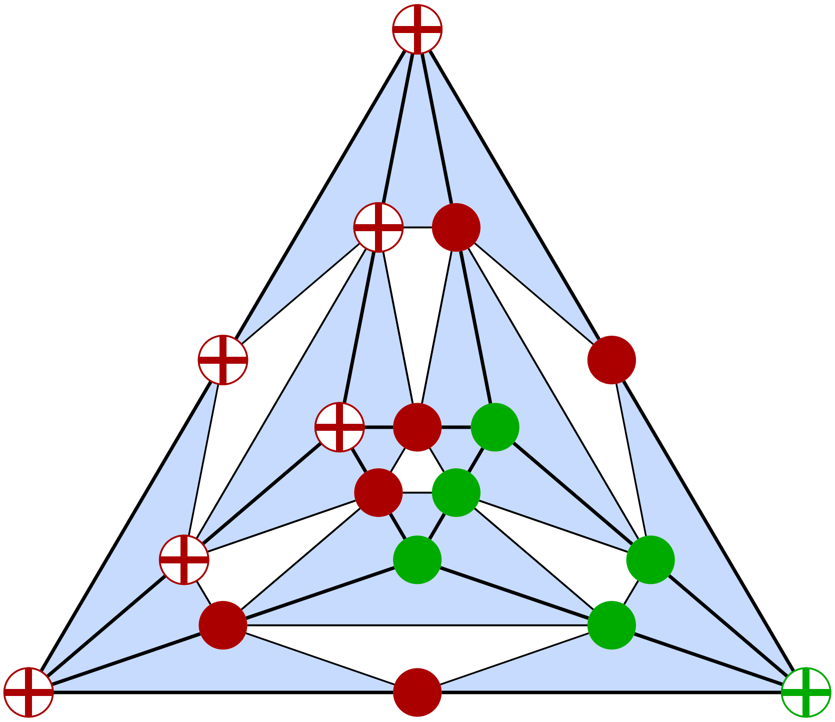

Figure 8. $\dim S_2^1 = 7$ on the symmetric Morgan-Scott split.

The geometry may also affect the Solvability of certain interpolation problems.

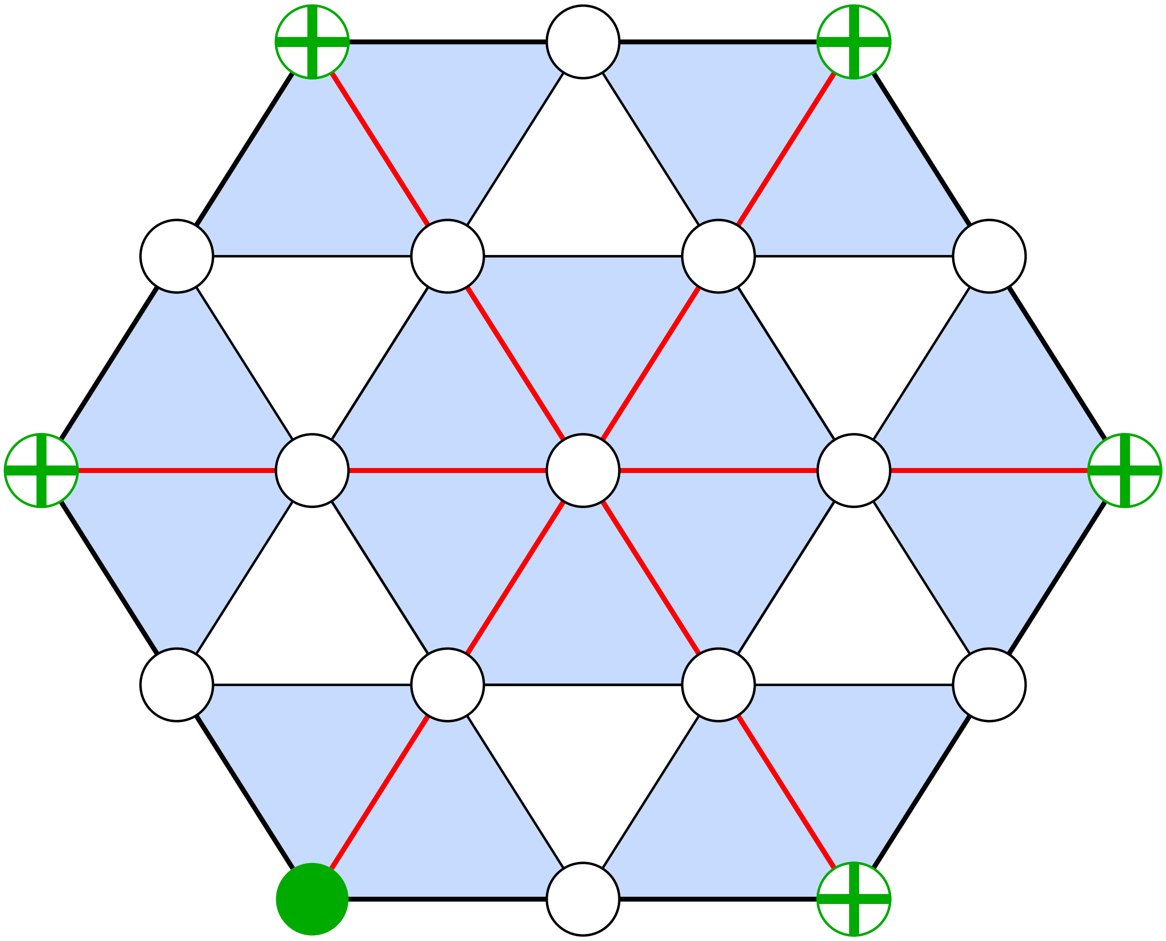

Figure 9. S_2^1 interpolation on a regular hexagon.

What's happening?

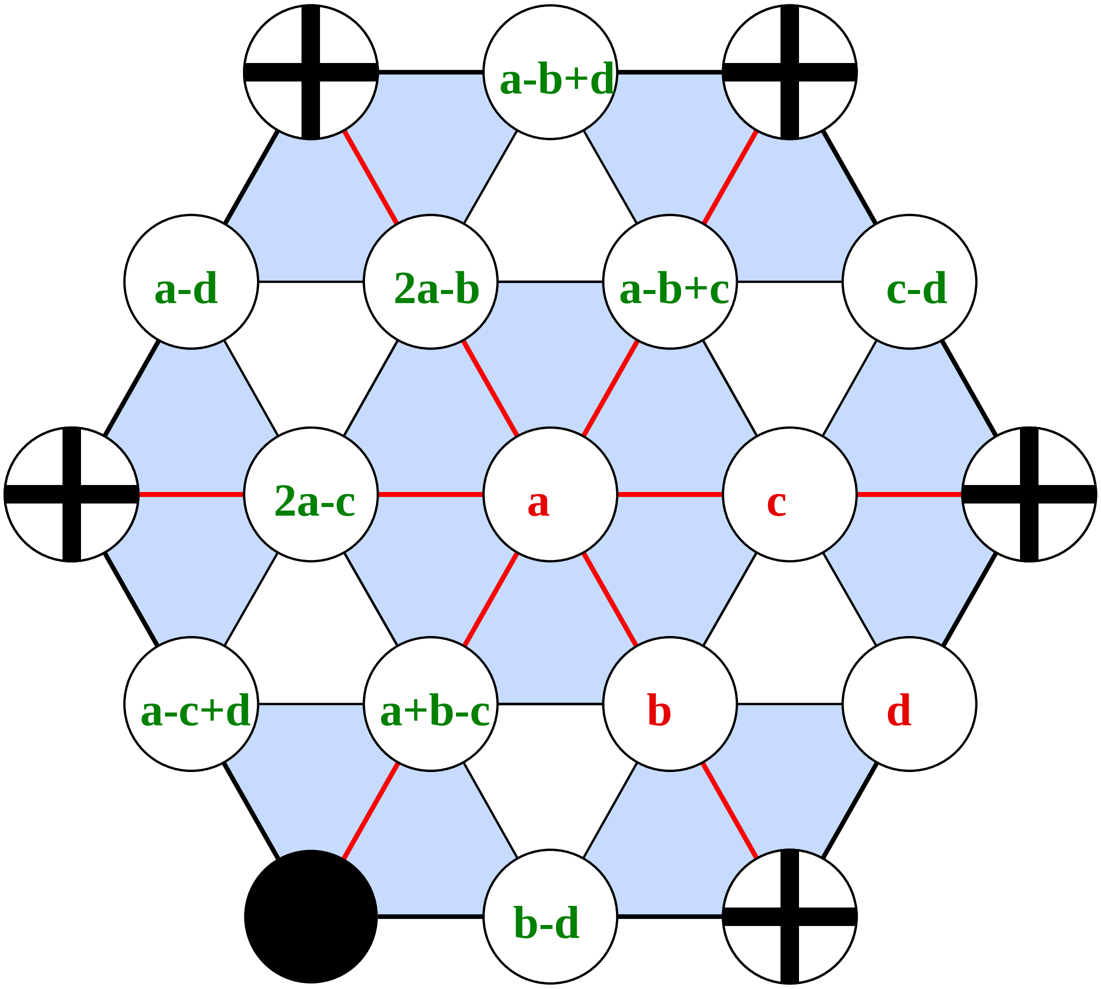

Figure 10. Reason for Failure.

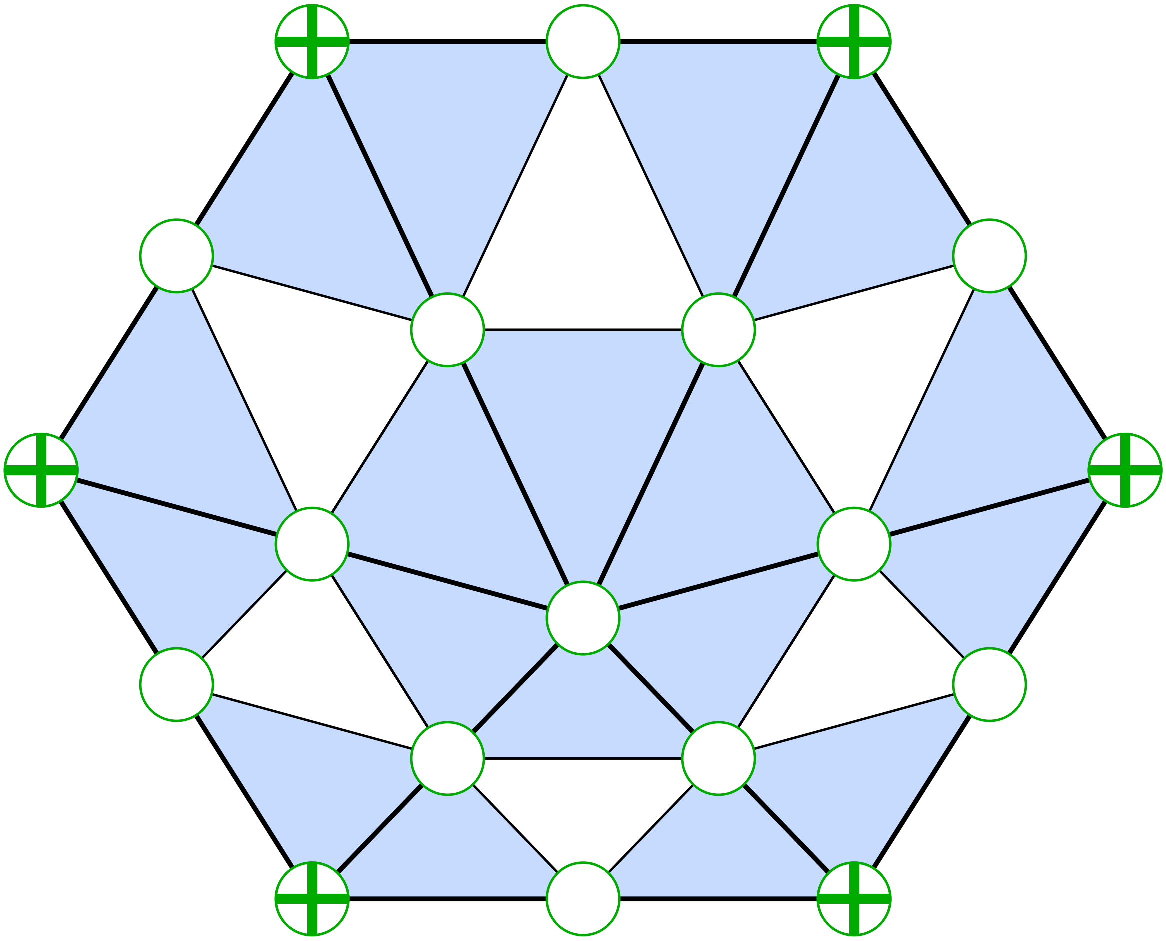

Figure 11. S_2^1 interpolation on a generic hexagon.

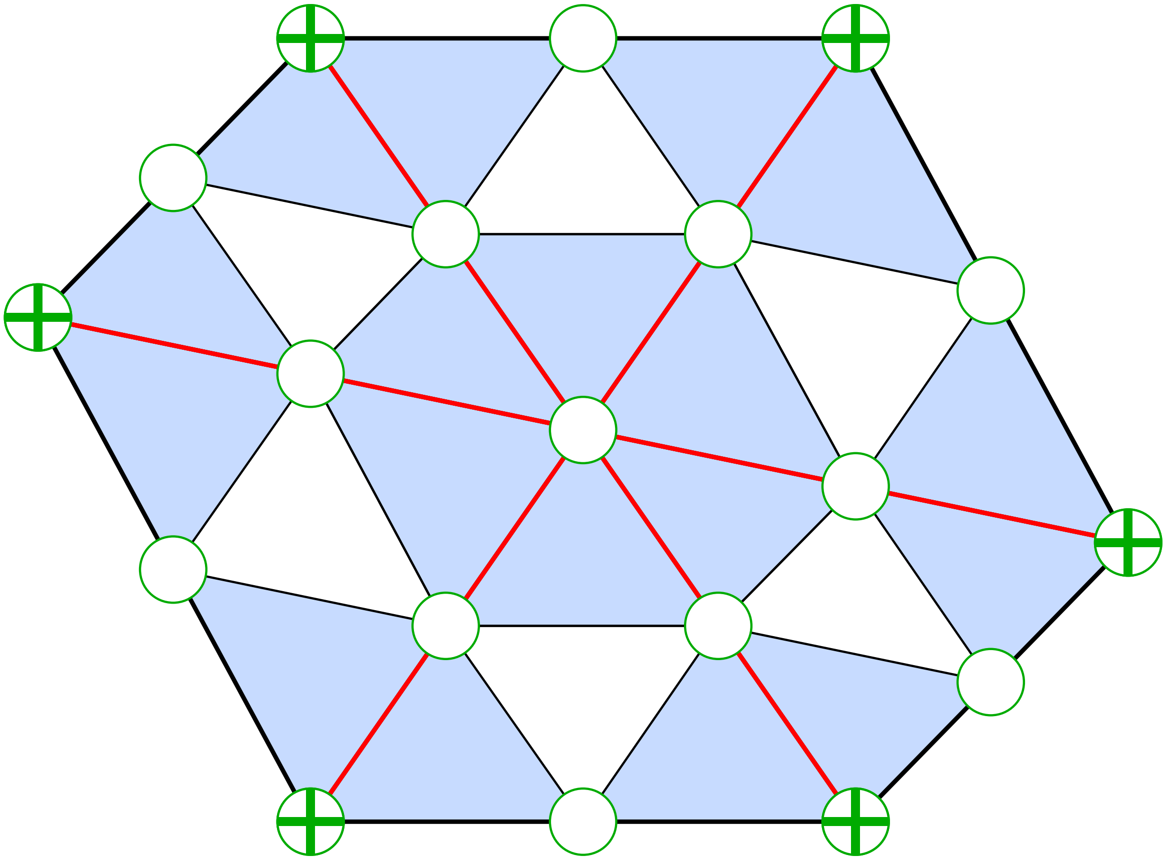

Figure 12. S_2^1 interpolation on a defective hexagon.

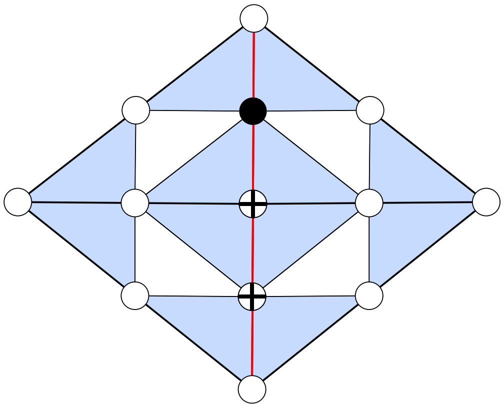

Figure 13. S_2^1: A peculiar effect.

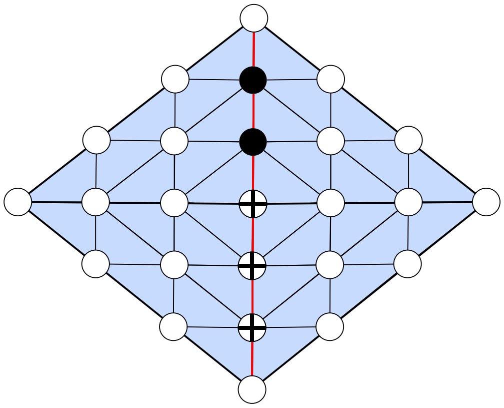

Figure 14. S_3^2: The same effect.

Smoothness conditions can have peculiar long range effects. The above examples are due to Tanya Sorokina.

This does not work on singular or generic degree 4 vertices!

Generic Dimension

Every spline space S has a generic dimension. If the dimension of S does not equal its generic value then there is an arbitrarily small perturbation of the location of the vertices such that the dimension of S does equal the generic value. Any other dimensions can only be larger than the generic dimension.

Proof sketch: Let $S = S_d^r$ be the subspace of S_d^0 with a coefficient vector c that satisfies the smoothness conditions

$$Ac = 0.$$

The entries of A are rational functions of the location of the vertices of the underlying triangulation.

Let D be the minimum dimension of S. Then $D = \mathrm{dim} S_d^0 - R$, with $R = \mathrm{rank} A$, and where (without loss of generality)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with A_{11} being a non-singular $R \times R$ matrix.

The expression $\det A_{11}$ is a non-zero rational function of the locations of the vertices, and can vanish only on a set of measure zero.

The generic dimension of S^1_4 is $6V-3$. Proof by Induction

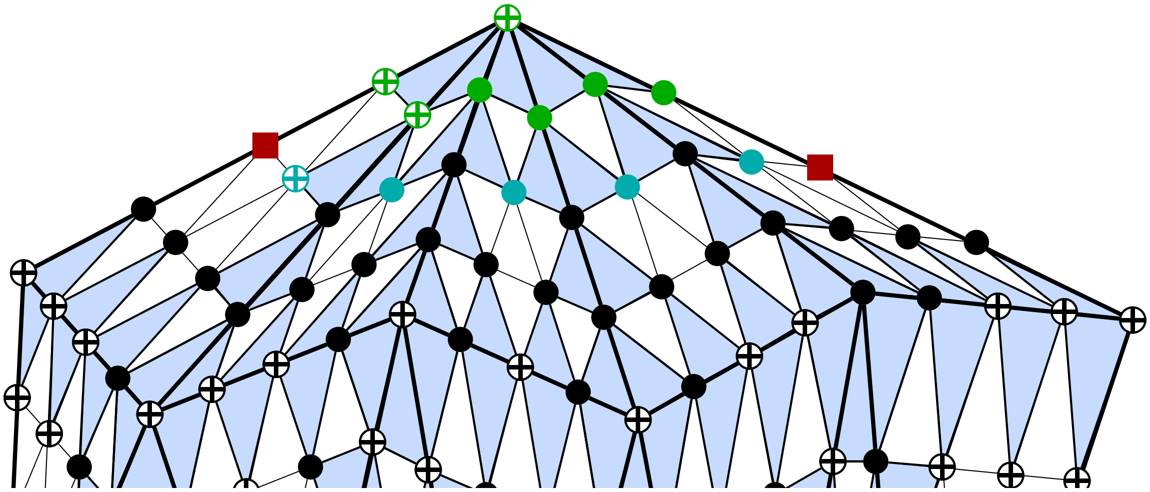

Figure 15. $\dim S^1_4 = 6V-3$.

Add the star of a boundary vertex to the growing triangulation T.

Black points are determined by the spline on T.

The green, red, and cyan points are newly imposed.

3 green, 2 red, and 1 cyan point are newly assigned.

Also shows that one can interpolate generically to function and gradient at vertices.

- Things get easier as the polynomial degree increases.
- Exact dimension known if $r = 1$ and $d \geq 4$, or $r > 1$ and $d \geq 3r + 2$.
- Generic dimension known for S_2^1 and S_3^1, and for $d = 3r + 1$ when $r > 1$.
- Dimensions and many other facts known on many types of special triangulations.
- Most famous outstanding problem:

  V_B: number of boundary vertices

  V_I: number of interior vertices

  σ: number of singular vertices

  Problem first mentioned in a conjecture by Gil Strang in the early seventies. What's the dimension of S_3^1?

  $$\boxed{\dim S_3^1 \geq 3V_B + 2V_I + 1 + \sigma}$$

  Does equality hold? Conjecture: yes.

  If you solve this problem I want to know about it!

Why is this so hard? You can’t localize things!

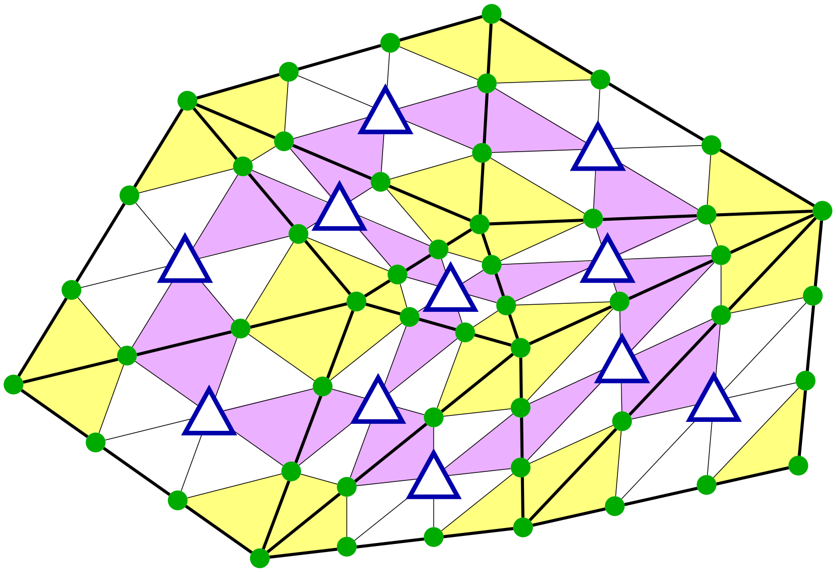

Figure 16. $\dim S_3^1$.

$$\dim S_3^1 \geq 3V + N - E_I + \sigma = 3V_B + 2V_I + 1 + \sigma.$$

What works for large d? The Morgan Scott idea.

Use vertex globs (green), edge globs (blue) , and inactive points (red). Requires $d > 4r$. Smoothness conditions decouple.

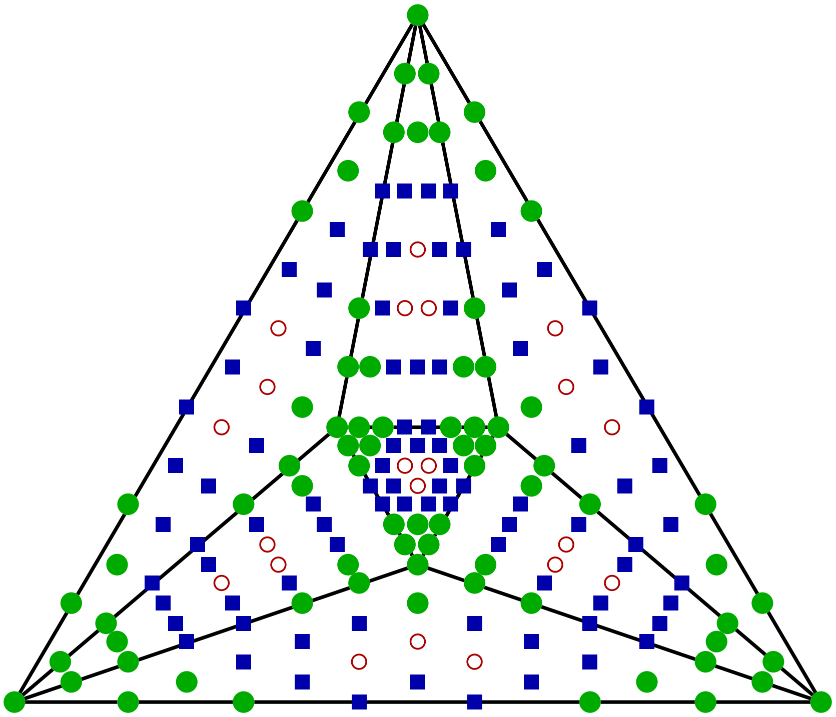

Figure 17. The Morgan-Scott idea, $r = 1$, $d = 6$.

Trivariate Splines

Consider a triangulation of a polyhedral domain by tetrahedra.

Similar definition as before.

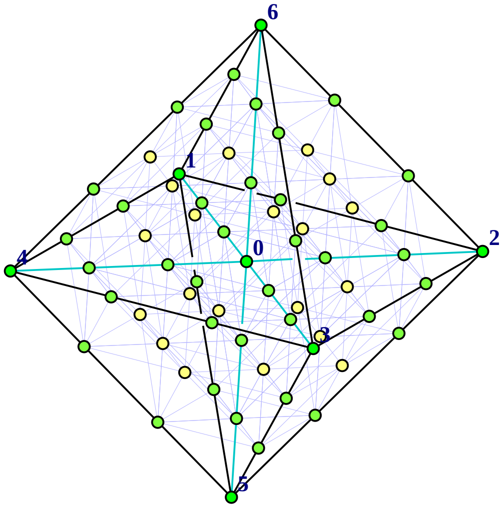

Figure 18. S_3^1 on an octahedron.

download software: www.math.utah.edu/~pa/3DMDS

- Similar problems. Dimension depends on the geometry.

- Morgan-Scott idea requires $d > 8r$.

- Generic dimension is known for $r = 1$ and $d \geq 8$.

- One new problem:

— Knowing the dimension of the trivariate space S_d^r for sufficiently large d means we know the dimension of the bivariate space **for all** d.

— To see this construct a three dimensional triangulation by starting with a planar triangulation T and then connecting every vertex of T to a new vertex in $\mathbb{R}^3$ outside of the plane containing T.

Coning

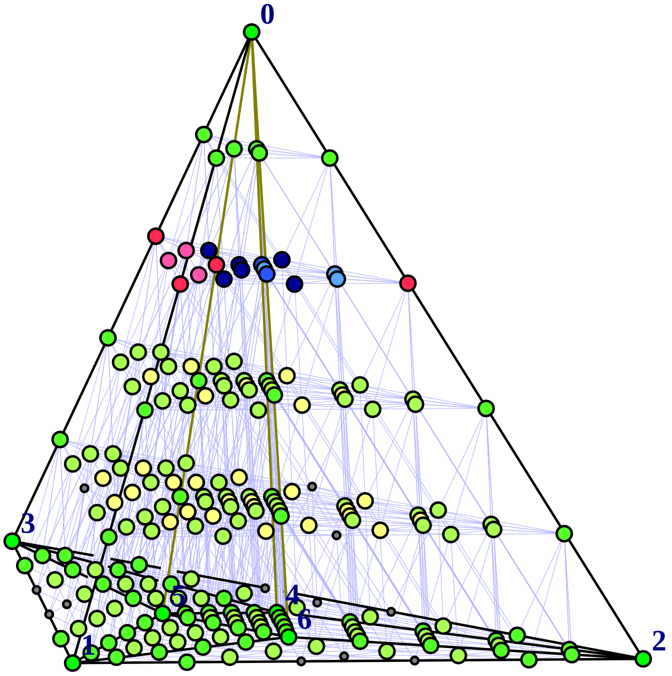

Figure 19. Coning.

Lift planar Triangulation to R^3. Smoothness conditions decouple.

Macro Elements

Most applications of multivariate splines are based on so called “macro elements” (when approximating data) or “finite elements” (when solving differential equations).

- The interpolant is determined on each simplex by data on that simplex.
- Simplices may be subdivided.
- The overall spline space is a sub or superspace of the full space S_d^r
- Many macro schemes are known in 2, 3, or n variables.

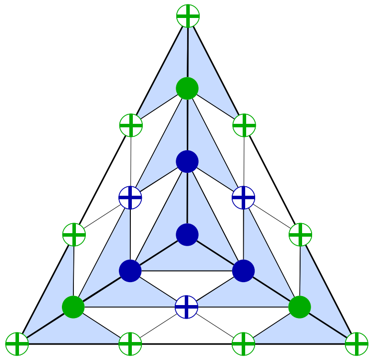

Figure 20. The planar Clough-Tocher element.

Recent Tetrahedral Macro Elements

Schumaker, Sorokina, Worsey: $r = 1$, $d = 2$, 504 microtetrahedra, no constraints, JAT, to appear.

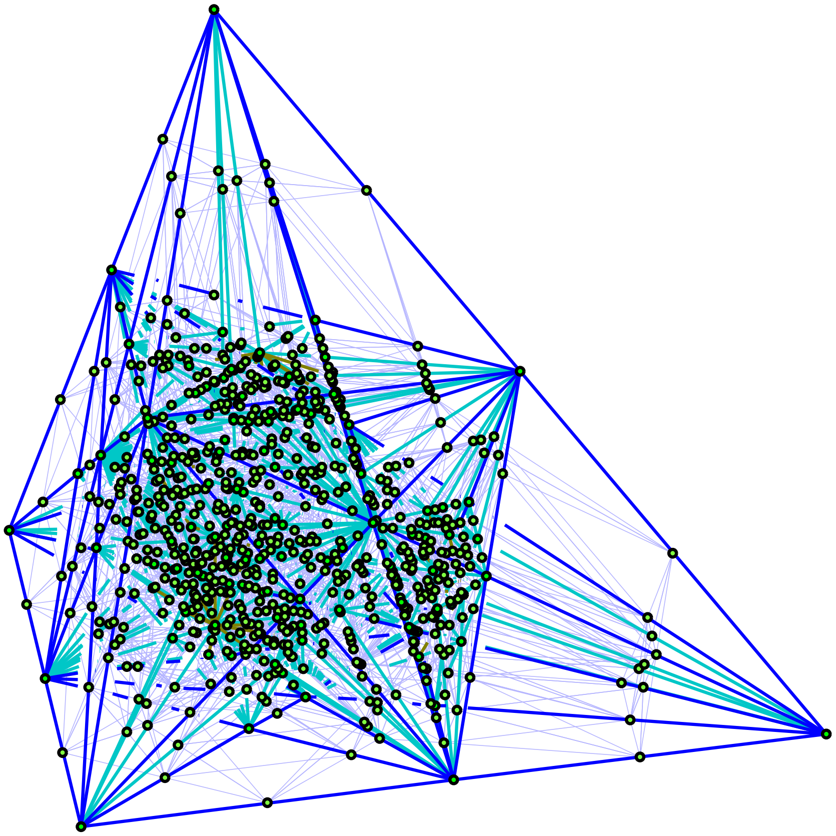

Figure 21. A quadratic macro element.

Alfeld and Sorokina. $r = 1$, $d = 3$, 60 microtetrahedra, no constraints, submitted for publication.

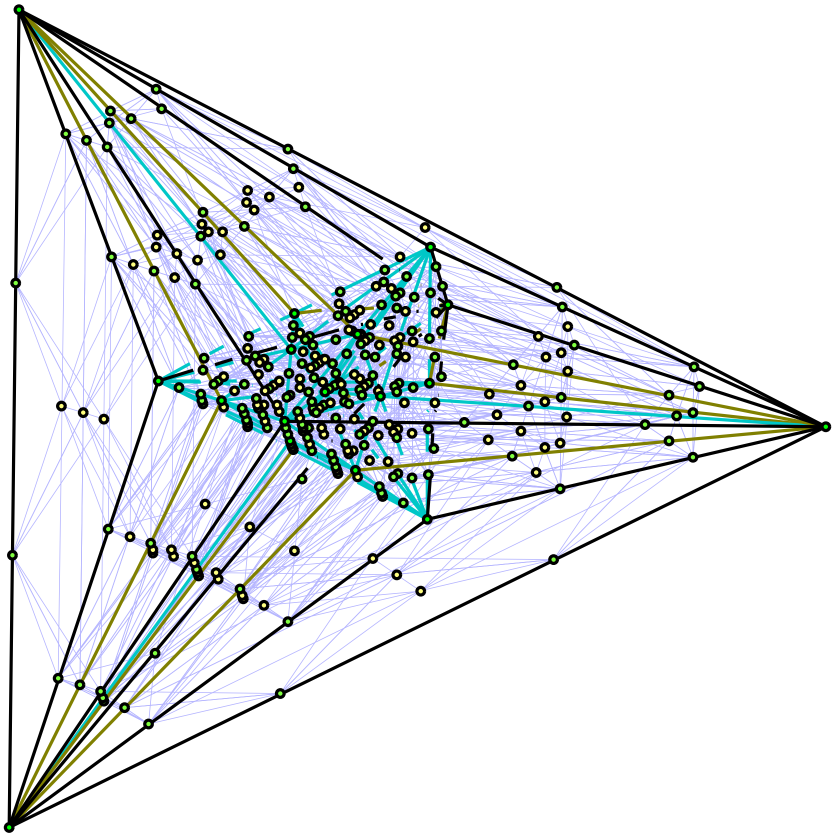

Figure 22. A cubic macro element.

Some Open Questions

2 Variables:

- What is the dimension of S_3^1?
- Can one interpolate to vertex data with S_3^1?
- What is the generic dimension of S_d^r, $r > 1$, $r < d \leq 3r$.
- What is the dimension of S_2^1? (Much harder than S_3^1)
- Can one interpolate with S_4^1 to function and gradient values at the vertices?

$k > 2$ variables

- More or less tight bounds on the dimensions
- For given r, what are minimum values of d, as a function of k, for S_d^r

— to have a local basis

— to able to interpolate to function values at vertices

— to be large enough to interpolate to function values at vertices.