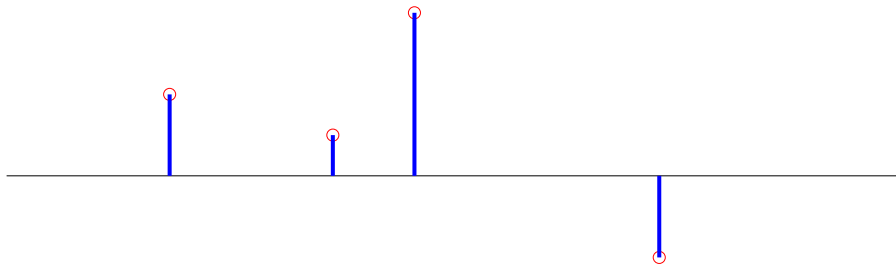


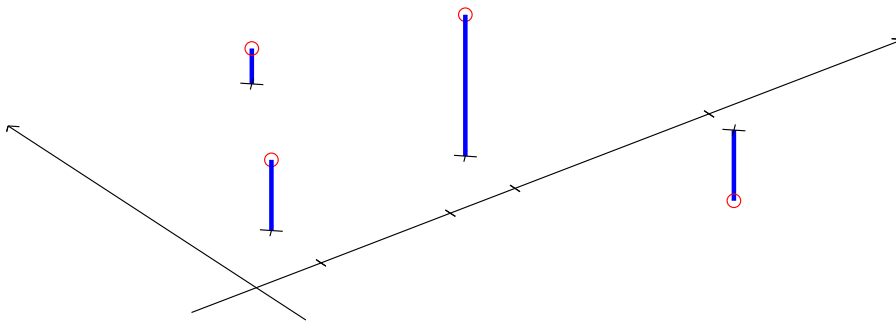
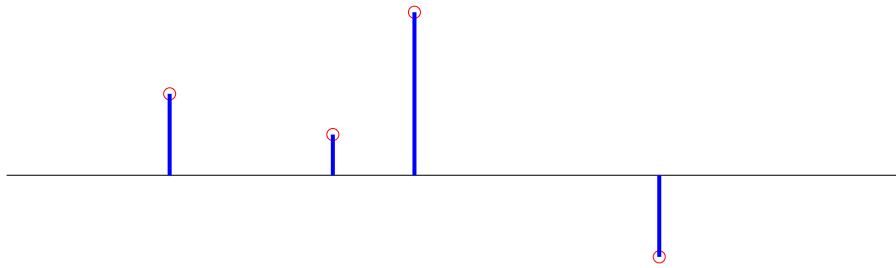
Issues in Multivariate Polynomial Interpolation

Carl de Boor

Polynomial interpolation

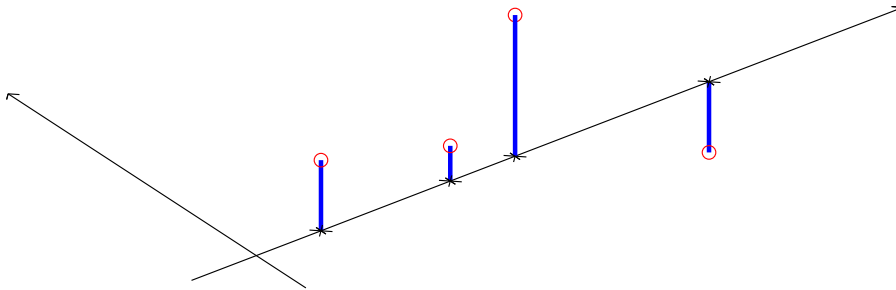
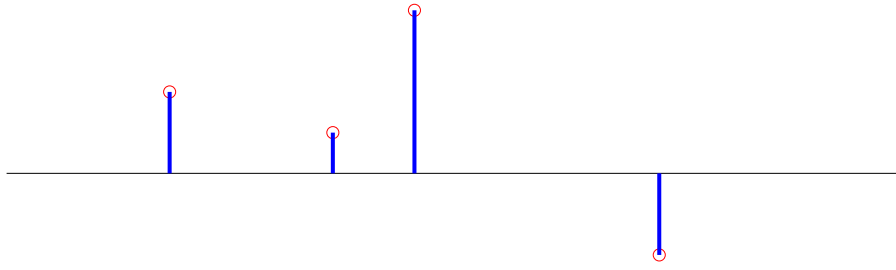


Multivariate polynomial interpolation



$$\dim \Pi_{\leq 1}(\mathbb{R}^2) = 3 < 4 < 6 = \dim \Pi_{\leq 2}(\mathbb{R}^2)$$

Multivariate polynomial interpolation



Now, even $\Pi_{\leq 2}(\mathbb{R}^2)$ won't do; need at least $\Pi_{\leq 3}(\mathbb{R}^2)$.

Multivariate polynomial interpolation

Basic Problem: Given a finite set \mathbf{T} of data **sites** τ in \mathbb{F}^d , how to choose a space F of polynomials so that, for every choice of data **values**

$$a = (a(\tau) : \tau \in \mathbf{T}) \in \mathbb{F}^{\mathbf{T}},$$

there is exactly one element $f \in F$ that matches this information, i.e., satisfies

$$f(\tau) = a(\tau), \quad \tau \in \mathbf{T}.$$

I.e., want F to be **correct for** \mathbf{T} .

unisolvant, poised, regular, ...



\mathbb{F} is \mathbb{R} or \mathbb{C} .

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I.e., want F to be **correct for** T . unisolvent, poised, regular, ... \downarrow

Equivalently, want the linear map \mathbb{F} is \mathbb{R} or \mathbb{C} .

$$\Lambda : g \mapsto g|_T \in \mathbb{F}^T$$

(of restriction to T) to be onto and 1-1 when restricted to F . Then

$$P_T := (\Lambda|_F)^{-1} \Lambda \quad : g \mapsto \sum_{\tau \in T} \ell_\tau g(\tau)$$

is the linear projector that associates each g with its unique interpolant $P_T g$ at T in F (and $(\Lambda|_F)^{-1}$ is the corresponding **Lagrange basis** for F).

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with

$$\text{Exp}(T) := \text{span}\{e_\tau : \tau \in T\},$$

$$e_\tau : x \mapsto e^{\tau^* x},$$

$$\tau^* x := \sum_{j=1}^d \overline{\tau(j)} x(j).$$

???

space of **formal power series**: $A_0 := \mathbb{F}[[x]] := \{g = \sum_{\alpha \in \mathbb{Z}_+^d} \widehat{g}(\alpha) (\)^\alpha\}$

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standard pairing: $A_0 \times \Pi : (g, p) \mapsto \langle g, p \rangle := \sum_{\alpha} \overline{\widehat{g}(\alpha)} \alpha! \widehat{p}(\alpha)$

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$$\sum_{\alpha} (\overline{\tau^\alpha} / \alpha!) x^\alpha = \sum_{\alpha} \overline{\tau^\alpha} x^\alpha / \alpha! = \sum_k (\tau^* x)^k / k! = \exp(\tau^* x) = e_\tau(x)$$

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$\lim_{h \rightarrow 0} g(h \cdot) / h^{\text{ord } g} = g^{[\text{ord } g]} =: g_\downarrow$, $\text{ord } g := \min\{|\alpha| : \widehat{g}(\alpha) \neq 0\}$
least form, initial form, ...

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$$g = \sum_k g^{[k]} = g_\downarrow + \text{h.o.t.}$$

Use Gram-Schmidt idea to derive, from the basis $(e_\tau : \tau \in T)$ for $\text{Exp}(T)$, a basis B for $\text{Exp}(T)$ for which $\langle b, c_\downarrow \rangle = \delta_{bc}$, $b, c \in B$. Then, for any p , $P_T p := \sum_{b \in B} \langle b, p \rangle b_\downarrow \in \Pi_T$ and $\text{Exp}(T) \perp p - P_T p$, hence $p = P_T p$ on T .

Note: $\deg b_\downarrow > \deg p \implies \langle b, p \rangle = 0$. Hence, $\deg P_T p \leq \deg p$.

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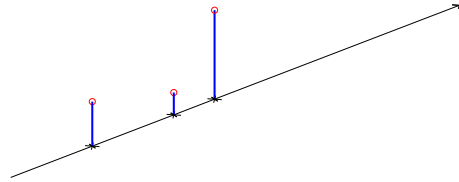
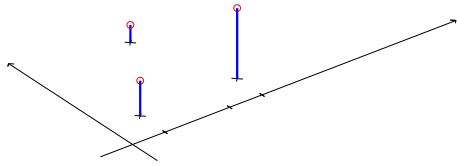
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- Π_T is easy to construct (by Gauss elimination by segments);
- ...

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Necessarily, $\dim F > n$ (loss of Haar). However, $F = \Pi_{<\#T}$ will always work:

$$I_T : p \mapsto \sum_{\tau \in T} p(\tau) \prod_{\sigma \in T \setminus \tau} \frac{(\tau - \sigma)^*(\cdot - \sigma)}{(\tau - \sigma)^*(\tau - \sigma)} \in \Pi_{<\#T}.$$

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But $\Pi_{<\#T}$ is far from “smallest”:

Example: $d = 2, \#T = 3$

$$\dim \Pi_{<3} = 6 > 4 = \dim(\Pi_{\leq 1} + \text{span}(()^{2,0} + ()^{0,2}))$$



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(the radical ideal with variety T), hence (for a “good” F) hope for

$$p(x) - P_T p(x) = \sum_{b \in B_T} b(x) \int M_b(x, \cdot) \tilde{b}_\uparrow(D) p$$

with

- B_T and $\{\tilde{b} : b \in B_T\}$ ‘nice’ generating sets for $\mathbf{ideal}(T)$,
- $b_\uparrow(D) := \sum_{|\alpha|=\deg b} \hat{b}(\alpha) D^\alpha, \quad D^\alpha := D_1^{\alpha(1)} \dots D_d^{\alpha(d)}.$

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“good” F ? Would need that $F = \bigcap_{b \in B_T} \ker \tilde{b}_\uparrow(D)$. Fortunately,

$$\Pi_T = \bigcap_{b \in B_T} \ker \bar{b}_\uparrow(D) .$$

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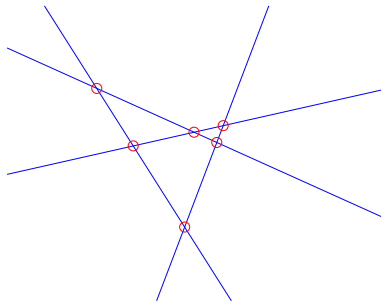
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For example, there is such a formula when $\mathbb{F} = \mathbb{R}$ and T is a **natural lattice**, i.e., (**Chung&Yao**) the collection of all intersections of d hyperplanes from a collection of hyperplanes in general position.



Sauer&Xu have a somewhat more complicated formula for any T correct for $\Pi_{\leq k}$ for some k .

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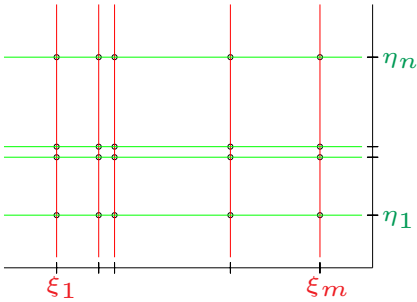
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Tensor product example:



$$T := ((\xi_i, \eta_j) : i = 1:m; j = 1:n)$$

$$p(x) - P_T p(x) = \prod_i (x(1) - \xi_i) \int M_1(x, \cdot) D_1^m p + \prod_j (x(2) - \eta_j) \int M_2(x, \cdot) D_2^n p$$

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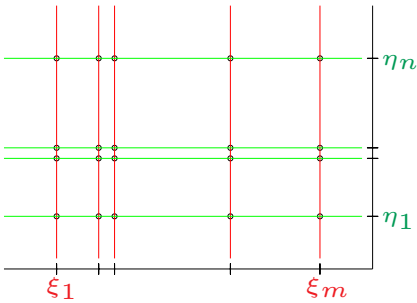
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Problem: What happens when, e.g., $\xi_i \rightarrow \xi_{i+1}$?

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If $Q := \lim_{T \rightarrow \Sigma} P_T$ exists, it is an **ideal projector**, i.e., $\ker Q = \text{ran}(\text{id} - Q)$ is an ideal.

Such Q must satisfy the identity

$$Q(fg) = Q(fQg), \quad f, g \in \Pi,$$

therefore (with $f = 1$), $Qg = Q^2g$, hence Q is a projector, in particular $\ker Q = \text{ran}(\text{id} - Q)$, while $Q(\Pi \cdot (\text{id} - Q)\Pi) = \{0\}$, i.e., $\text{ran}(\text{id} - Q)$ is an ideal.

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Boris Shekhtman, via Geir Ellingsrud: yes for $d = 2$, no for $d > 2$.

closely related question: Can any sequence (M_1, \dots, M_d) of (complex-valued) commuting matrices be approximated by such a sequence of **diagonalizable ones**?

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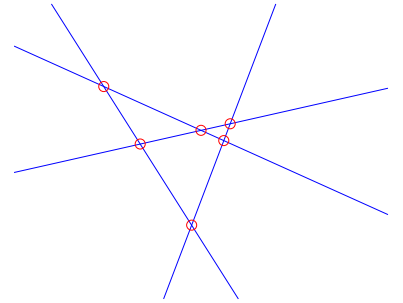
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Chung-Yao'77 recipe: Choose T with $\dim \Pi_{\leq k} \leq \#T$ so that, for each $\tau \in T$, $T \setminus \{\tau\}$ lies in a union of $\leq k$ hyperplanes that do not contain τ . Such T is called a **GC_k-set** (short for a **set satisfying the Geometric Conditions of degree k**) and is k -correct.



E.g., Chung-Yao's natural lattice.

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Why does it work?

\implies . For each $\tau \in T$, have a product of $\leq k$ polynomials of degree 1 that vanishes on $T \setminus \tau$ but not on τ .

\implies . $\Pi_{\leq k} \rightarrow \mathbb{F}^T : p \mapsto p|_T$ is onto, while $\dim \Pi_{\leq k} \leq \#T = \dim \mathbb{F}^T$.

\implies . $\Pi_{\leq k} \rightarrow \mathbb{F}^T : p \mapsto p|_T$ is invertible. In particular, $\dim \Pi_{\leq k} = \#T$.

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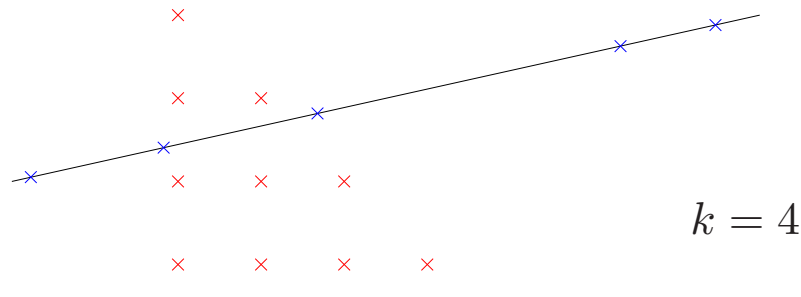
Chung-Yao'77 recipe: Choose T with $\dim \Pi_{\leq k} \leq \#T$ so that, for each $\tau \in T$, $T \setminus \{\tau\}$ lies in a union of $\leq k$ hyperplanes that do not contain τ . Such T is called a GC_k -set (short for a **set satisfying the Geometric Conditions of degree k**) and is k -correct.

Gasca et alii (e.g., Maeztu, Carnicer, Pēna, Sauer, ...): thorough study for $d = 2$ and much work for $d > 2$.

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(Bercolari'14) Radon'48 recipe: If P is $(k-1)$ -correct and, for some hyperplane $H \subset \mathbb{F}^d \setminus P$, Σ is correct for $\Pi_{\leq k}(H)$, then $P \cup \Sigma$ is k -correct.



Lagrange vs Newton ??

Call a hyperplane H **maximal** for a k -correct T if $T \cap H$ is correct for $\Pi_{\leq k}(H)$.

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Gasca-Maeztu Conjecture (1982). *Any bivariate GC_k -set has at least one maximal. (Proven for $k < 5$)*

Carnicer-Gasca. *If, for some n , every bivariate GC_k -set with $k \leq n$ has one maximal, then every such GC_k -set has at least three maximals.*

$d > 2$???

FINIS

standard pairing: $A_0 \times \Pi : (g, p) \mapsto \langle g, p \rangle := \sum_{\alpha} \overline{\widehat{g}(\alpha)} \alpha! \widehat{p}(\alpha) = p(D) \overline{g}(0)$

[BR91-92]. $\Pi_{\mathbb{T}} = \cap_{p|_{\mathbb{T}}=0} \ker \overline{p}_{\uparrow}(D)$.

Proof:

$$\begin{aligned} p|_{\mathbb{T}} = 0 &\implies \text{Exp}(\mathbb{T}) \perp p \\ &\implies \Pi_{\mathbb{T}} \perp p_{\uparrow} & \langle g, p \rangle = 0 &\implies \langle g_{\downarrow}, p_{\uparrow} \rangle = 0 \\ &\implies \forall \{g \in \Pi_{\mathbb{T}}\} \overline{p}_{\uparrow}(D)g(0) = 0 \\ &\implies \forall \{g \in \Pi_{\mathbb{T}}\} \forall \{\alpha\} \overline{p}_{\uparrow}(D)D^{\alpha}g(0) = 0 \\ &\implies \forall \{g \in \Pi_{\mathbb{T}}\} \overline{p}_{\uparrow}(D)g = 0 \end{aligned}$$

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 \end{aligned}$$

Hence $\Pi_{\mathbb{T}} \subseteq K$.

$$\begin{aligned}
 \dim \Pi_{\mathbb{T}} < \infty &\implies k := \max\{\deg f : f \in \Pi_{\mathbb{T}}\} < \infty \\
 &\implies \forall \{|\alpha| = k + 1\} (())^{\alpha} - P_{\mathbb{T}}(())^{\alpha}_{\uparrow} = (())^{\alpha} \\
 &\implies K \subset \cap_{|\alpha|=k+1} \ker D^{\alpha} = \Pi_{\leq k} \subset \Pi
 \end{aligned}$$

Hence, $g \in K$ implies that $\Pi \ni p := g - P_{\mathbb{T}}g \in K \subset \ker \overline{p}_{\uparrow}(D)$, i.e., $\overline{p}_{\uparrow}(D)p = 0$, hence $p = 0$, so $g \in \Pi_{\mathbb{T}}$.