Macro Elements

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Second International Workshop on Algebraic Geometry and Approximation Theory, Towson University, April 9–11, 2009

Thanks to Tanya Sorokina and Luis Garcia for organizing this conference, to Towson University for financing it, and to Raouf Boules for hosting it! Thanks for inviting me!

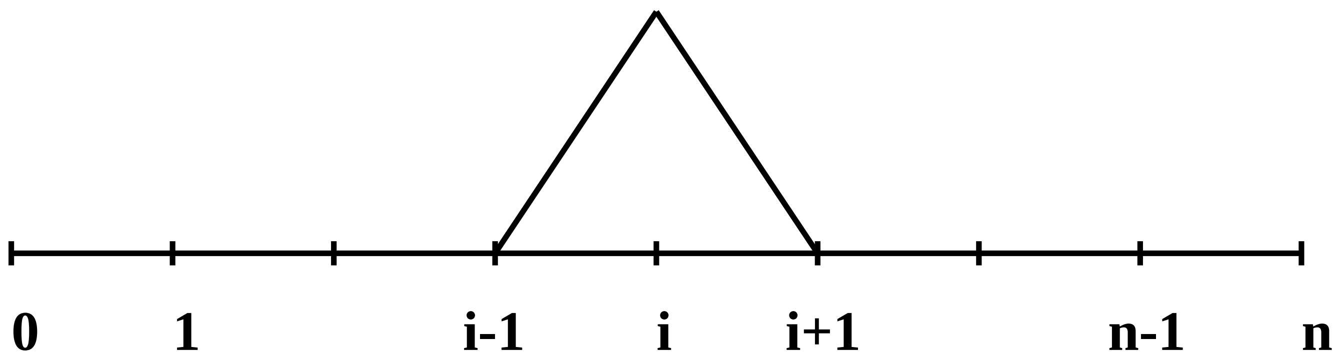

Figure 1. 1D Hat Functions.

$$h = \frac{1}{n}, \qquad x_i = ih, \quad i = 0, \ldots, n$$

$$\psi_i(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{if } x = x_j,\ j \neq i \\ \text{linear} & \text{in each } [x_{j-1}, x_j] \end{cases}$$

piecewise linear, globally continuous

Interpolation — Macro Element

Given (x_i, y_i), $i = 0, \ldots, n$

$$s(x) = \sum_{i=0}^{n} y_i \psi_i(x).$$

Variational Principle — DE — Finite Element

$$I(u) = \int_0^1 \big(u'(x)\big)^2 + \big(u(x)\big)^2 \mathrm{d}x = \min, \quad u(0) = u(1) = 1.$$

Gives rise to the Euler equation: $u'' = u$

Let $y_0 = y_n = 1$, find

$$y = [y_1, \ldots, y_{n-1}]$$

such that $I(s) = \min$.

Gives rise to a linear system

$$Ay = b.$$

Focus on Macro Elements

Properties:

- s is piecewise polynomial
- s is smooth (C^0 in this case)
- on each subinterval, s is determined uniquely by data on that subinterval.
- The (nodal) data y_i are the coefficients of the interpolant. The interpolant is in *cardinal form*.
- The support of the basis functions is small (as small as it can be — just two subintervals).
- Making a local change in the data affects the interpolant only locally.
- If the data are values of a linear function then the interpolant equals that linear function. (This is called *linear precision*, and its the key to error analysis.)
- These ideas can be generalized to arbitrary degrees of smoothness, using higher polynomial degrees.
- Our interest is in doing this in two or more variables (rather than just 1).

Triangulations

- What about two variables?
- Need to partition a polygonal domain
- rectangular grid leads to tensor product schemes. They form a huge area. But they are essentially univariate splines done twice.
- If points are distributed irregularly then **triangles** work.

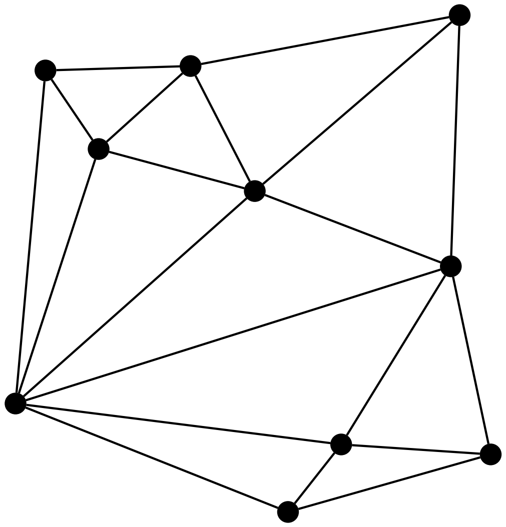

Figure 2. A triangulation.

Polynomials

Let Ω be the union of the triangles T_i, $i = 1, \ldots, N$, and P^d the $\binom{d+2}{2}$-dimensional space of bivariate polynomials of degree d.

$$P_d = \left\{ p : p(x_1, x_2) = \sum_{i=0}^{d} \sum_{j=0}^{d-i} \alpha_{ij} x_1^i x_2^j \right\}.$$

$$S_d^r = \left\{ s \in C^r(\Omega) : s|_{T_i} \in P^d \right\}.$$

The Bernstein-Bézier Form

Let Δ be a triangle with vertices v_1, v_2, and v_3.

For $x \in \mathbb{R}^2$, define its *barycentric coordinates*:

$$x = \sum_{i=1}^{3} b_i v_i \quad \text{where} \quad \sum_{i=1}^{3} b_i = 1.$$

Barycentric coordinates are linear functions of x.

Any polynomial $p \in P_d$ can be written uniquely in its *Bernstein-Bézier form* as:

$$p(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k.$$

where

c_{ijk} : *Bézier ordinates*

$P_{ijk} = \dfrac{iv_1 + jv_2 + kv_3}{d} \in \mathbb{R}^2$: *Domain Points*

$(P_{ijk}, c_{ijk}) \in \mathbb{R}^3$: *Bézier control points*

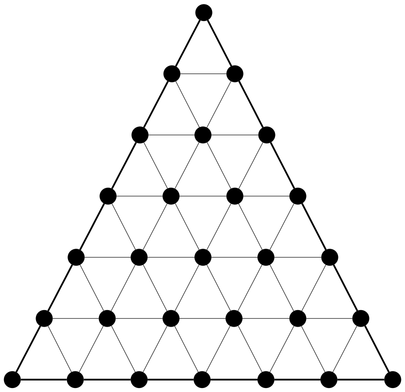

Figure 3. Domain points, $d = 6$.

Simple Geometry

- The control points at the vertices lie on the graph of the polynomial. This is because

$$p(v_1) = c_{d00}, \quad p(v_2) = c_{0d0}, \quad p(v_3) = c_{0d0}$$

- The control points in the 1-disk, i.e.,

$$(P_{d00}, c_{d00}), \quad (P_{d-1,1,0}, c_{d-1,1,0}), \quad (P_{d-1,0,1}, c_{d-1,0,1})$$

  lie in the tangent plane of p at v_1. Similarly for v_2 and v_3.

- The control points along an edge determine the values of the polynomial along that edge.
- The control points along an edge, and in the first row parallel to the edge, determine the values of first derivatives of the polynomial along that edge.

Smoothness Conditions

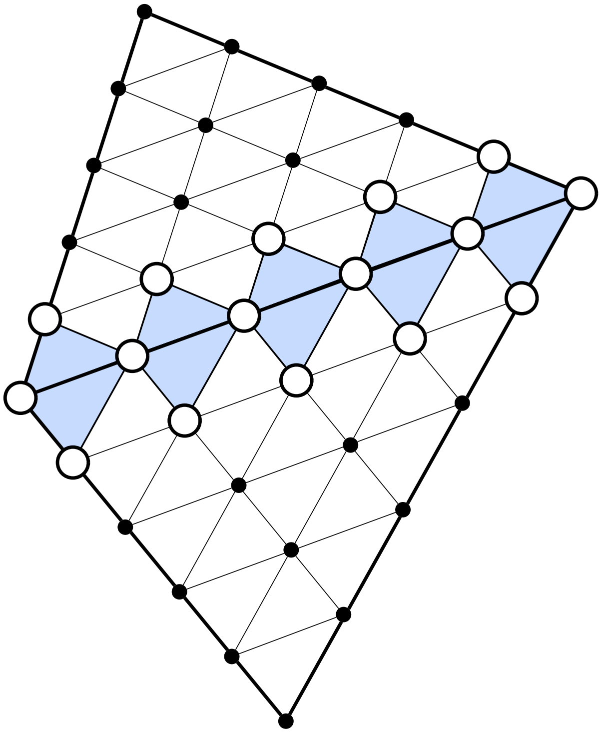

Figure 4. Smoothness conditions.

C^0: Control points along edges coincide

C^1: Quadrilaterals across edges are planar in $\mathbb{R}^3$.

C^r: Evaluate subpolynomials.

Courant Hat Functions (1943)

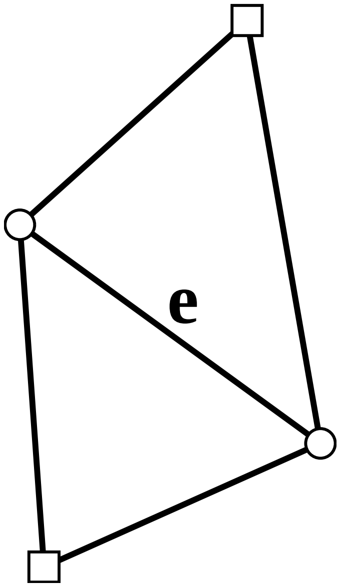

Figure 5. S_1^0 element.

- Space: S_1^0
- Nodal Data: function values at vertices.
- Those happen to equal the Bézier coefficients at the vertices.

Continuity

- linear functions on two neighboring triangles join continuously.
- Why?
- The restriction of each of the two bivariate linear functions to the common edge e gives a univariate linear function on the edge.
- That linear function is determined uniquely by two function values at the ends of the edge.
- Those data are common to the two triangles.
- The two restrictions are identical.
- The overall function is continuous.
- **Similar properties as in the univariate case:**
  - On each triangle the interpolant is determined by data on that triangle.
  - Local changes have local effects.
  - The scheme has linear precision

Beyond Continuity

- f a function of two variables
- We require continuity of derivatives across edges.
- Nodal values are function values and values of cartesian derivatives, e.g., gradients and Hessians.

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix}, \quad \nabla^2 f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

- **Directional derivatives**. $P \in \mathbb{R}^2$ a point and $e \in \mathbb{R}^2$ a direction.

$$\frac{\partial f}{\partial e}(P) = \frac{\mathrm{d}}{\mathrm{d}t} f(P + te)\bigg|_{t=0} = \nabla f(P) \circ e.$$

- Directional derivatives are not (usually) normalized, and are frequently in the directions of edges, or perpendicular to those.
- The significance of perpendicular directions is that they are defined by the edge only, and independent of the shape of the triangle.
- Two first order directional derivatives determine the gradient, three second order directional derivatives determine the Hessian.

Polynomial C^1 element (Zlamal, 1968)

- Space: subspace of S_5^1
- 21 Data:

– Function values, gradients, Hessians at vertices.

– Value of `pcbd`s at midpoints of edges.

$$f(v_i), f_x(v_i), f_y(v_i), f_{xx}(v_i), f_{xy}(v_i), f_{yy}(v_i),$$
$$\frac{\partial f}{\partial (v_i - v_j)^\perp}\left(\frac{v_i + v_j}{2}\right).$$

– **Stencil:**

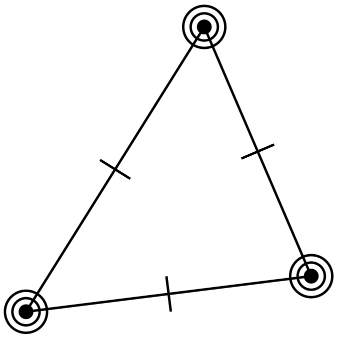

Figure 6. Stencil for S_5^1 element.

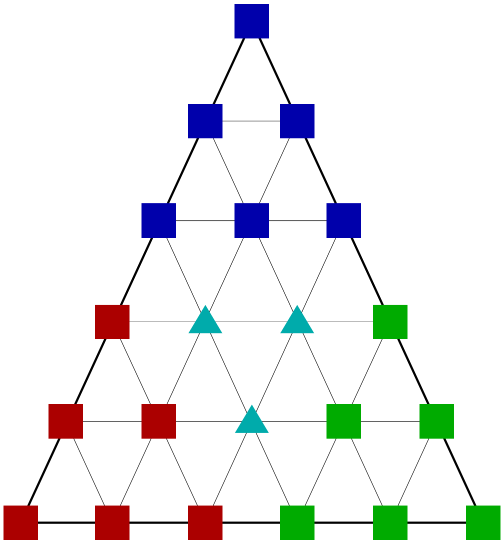

Figure 7. Bézier net for S_5^1 element.

- Red, green, and blue each go with one of the vertices.
- cyan triangles go with the edge conditions.
- The data determine all of the Bézier ordinates.

- **It's C^0.** Consider an edge shared by two triangles.
- Along a common edge, the restriction of each of the bivariate quintics is a univariate quintic.
- That quintic is determined uniquely by its value and its first and second derivatives (in the direction of the edge) at the two endpoints of the edge.
- These data are common to the two triangles.
- The univariate quintics are identical.
- **It's C^1.** It's enough to consider continuity of the `pcbd`.
- On each triangle that's quartic, and its restriction to the common edge is a univariate quartic.
- That quartic is determined by five data: the value of the quartic at each endpoint, the directional derivative in the direction of the edge at each endpoint, and the value at the midpoint.
- Those data are common to the two triangles, the `pcbd` is continuous.

Precision and Condensation of Parameters

- The macro scheme has quintic precision: A quintic is in the interpolating space, it interpolates to itself, and the interpolant is unique.
- On the other hand, those `pcbd`s are a nuisance.
- A user might be hard pressed to provide them (and might not even know about the underlying triangulation).
- Replace them with the requirement that `pcbd`s along edges *be cubic instead of quartic.*

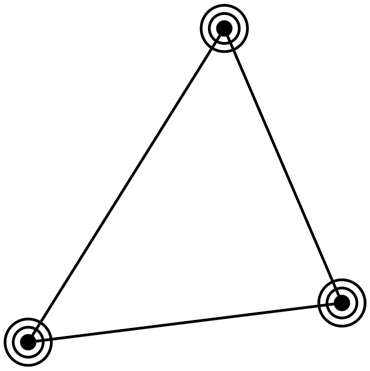

Figure 8. Stencil for condensed S_5^1 element.

- The resulting scheme has only quartic precision.

The Clough-Tocher Element (1965)

- It's a disadvantage to have to provide high degree derivatives.
- High polynomial degree may also be a disadvantage.
- Idea: **Subdivide the triangle**
- Clough-Tocher use S_3^1 on each triangle.
- Is globally C^1
- Stencil:

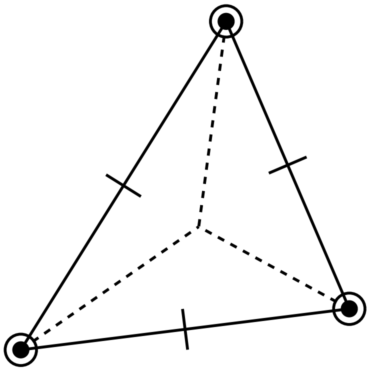

Figure 9. Stencil for Clough-Tocher Scheme.

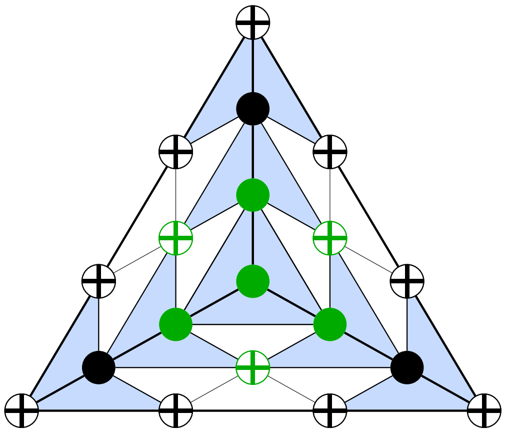

Figure 10. Bézier Net for Clough-Tocher Scheme.

- Plus signs in circles are determined by the data.
- Filled in circles are determined by smoothness conditions.
- Black points go with vertex data.
- Green points go with edge data.
- The scheme has cubic precision
- Parameter can be condensed: require that `pcbd`s be linear instead of quadratic.
- The condensed scheme has quadratic precision.

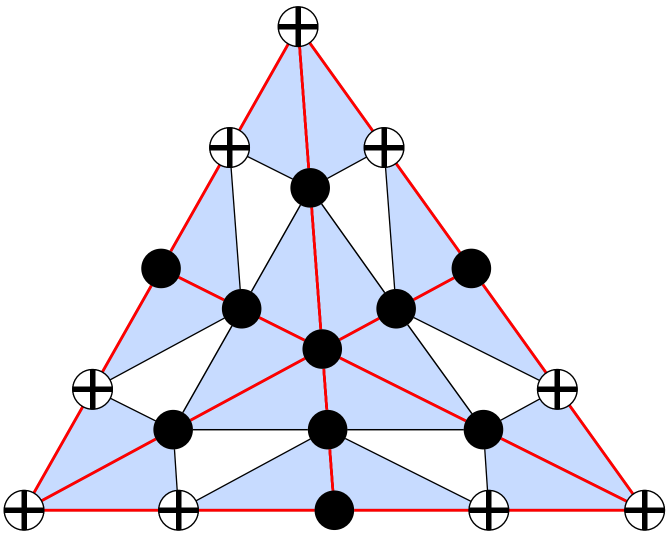

Figure 11. Bézier Net for the Powell-Sabin 6 split (1977).

- Is in S_2^1 on each triangle.
- It is globally C^1
- Has quadratic precision.
- does not require `pcbd` data.

Geometric Constraints

- **But:** The split points of interior edges must be such that the two line segment to the centroids are in the same line.
- Thus the line connecting two split points of neighboring triangles must intersect the common edge in its interior.
- This can always be accomplished, e.g., by picking the interior split point of each triangle to be its incenter.

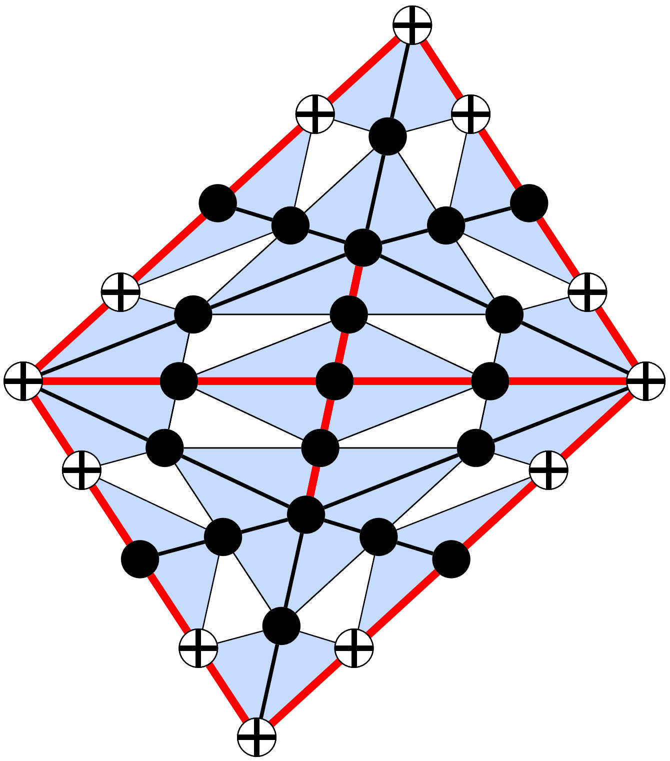

Figure 12. Correct Geometry for Powell Sabin 6.

- Circled Plus signs correspond to coefficients obtained by interpolation.
- Solid circles correspond to coefficients implied by smoothness conditions.

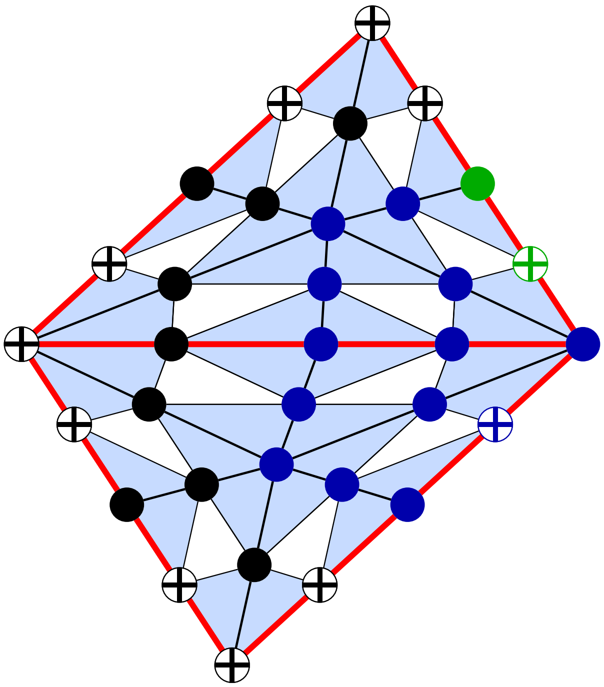

Figure 13. Incorrect Geometry for Powell Sabin 6.

- Only 11 data can be imposed.
- Assign black plus signs, implies black filled circles.
- Assigning the green circled plus point implies the solid green point.
- Assigning the blue circled plus determines all remaining coefficients (blue solid circles).

- Constraints can be eliminated by having more triangles.

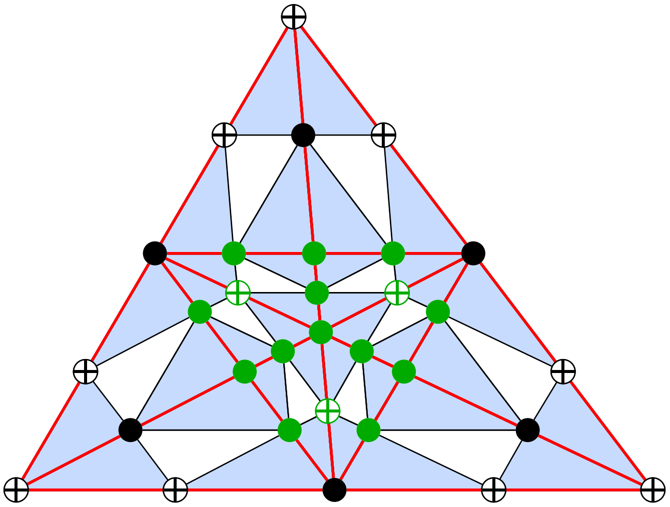

Figure 14. Bézier Net for the Powell-Sabin 12 split (1977).

- Has the same stencil as Clough-Tocher
- Is in S_2^1 on each triangle.
- Is globally C^1
- Has quadratic precision.
- Can be condensed by requiring that `pcbd`s be *linear instead of piecewise linear.*
- The condensed split *also has quadratic precision.*
- However, it does not require `pcbd` data.

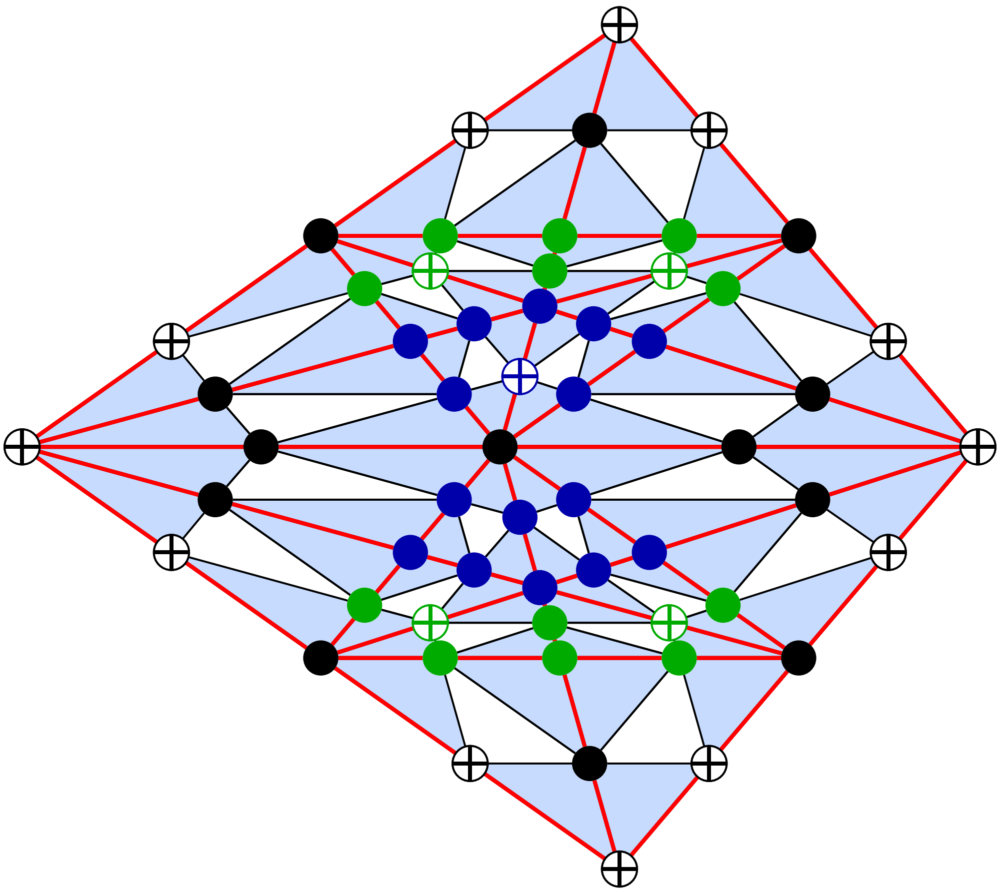

Figure 15. Powell-Sabin 12 split on two neighboring triangles.

- Split points of triangles and common edge are not aligned!
- It works!

Beyond Two Variables

- Many more schemes, and families of schemes with arbitrary smoothness, exist in the two variable case.
- But: move on to three or more variables.

$$\text{intervals} \longrightarrow \text{triangles} \longrightarrow \text{tetrahedra} \longrightarrow \text{simplices}$$

- A simplex in $\mathbb{R}^k$ is the convex hull of $k+1$ points (in general position).
- A polynomial of degree d in k variables has $\binom{d+k}{k}$ coefficients.
- The B-form works on simplices.
- The S_1^0 scheme works for all k.
- The minimum polynomial degree for a C^r scheme on an undivided simplex is
$$d = 2^{rk} + 1$$
- To reduce the polynomial degree we must divide the tetrahedron or simplex. The smaller the degree the larger the number of micro tetrahedra. How far will we go?

Trivariate Clough-Tocher, (Alfeld, 1984)

- $d = 5$ (instead of 9).
- split about the centroid
- require C^2 at the four boundary vertices (supersmoothness).

$$\mathrm{dim} S_5^1 = 68$$

- 40 Data: function, gradient, and Hessian values at boundary vertices.
- Require that perpendicular cross-boundary derivatives along boundary edges be cubic (instead of quartic)
- Require that perpendicular derivatives across faces equal a specific reduced cubic.
- Minimize certain fourth order derivatives at the centroid.
- Scheme has cubic precision. (This is the maximum degree if we are using only vertex data.)

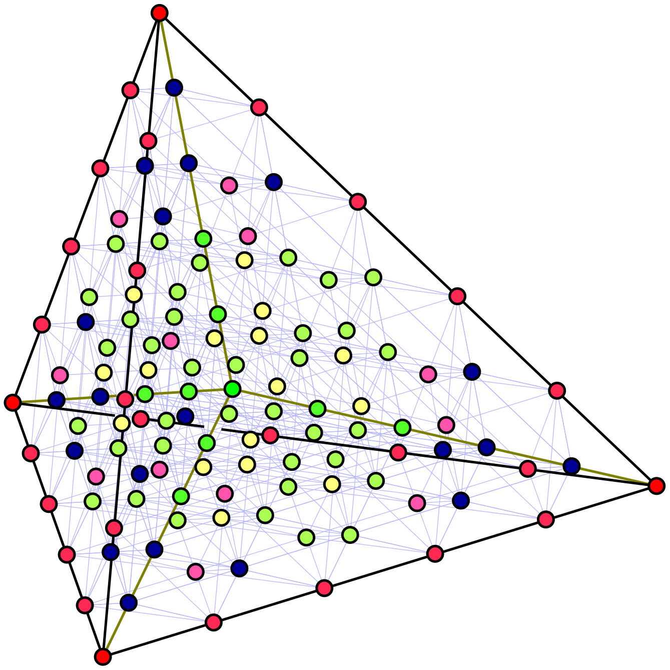

Figure 16. 3D-Clough-Tocher Scheme.

- red: defined by vertex data
- blue: implied by vertex data
- green and yellow: condensed

Worsey-Farin (1987)

- Apply CT on faces
- Connect to interior point of tetrahedron
- Get 12 micro tetrahedra
- $r = 1$, $d = 3$
- 28 data, function and gradient at vertices, derivatives at edges.
- cubic precision
- Edge data can be condensed, quadratic precision
- In the case $k = 3$ lines connecting split points of neighboring tetrahedra must intersect the common face.
- Generalizes to $k > 3$, but there are severe additional geometric constraints (Sorokina, 2008).

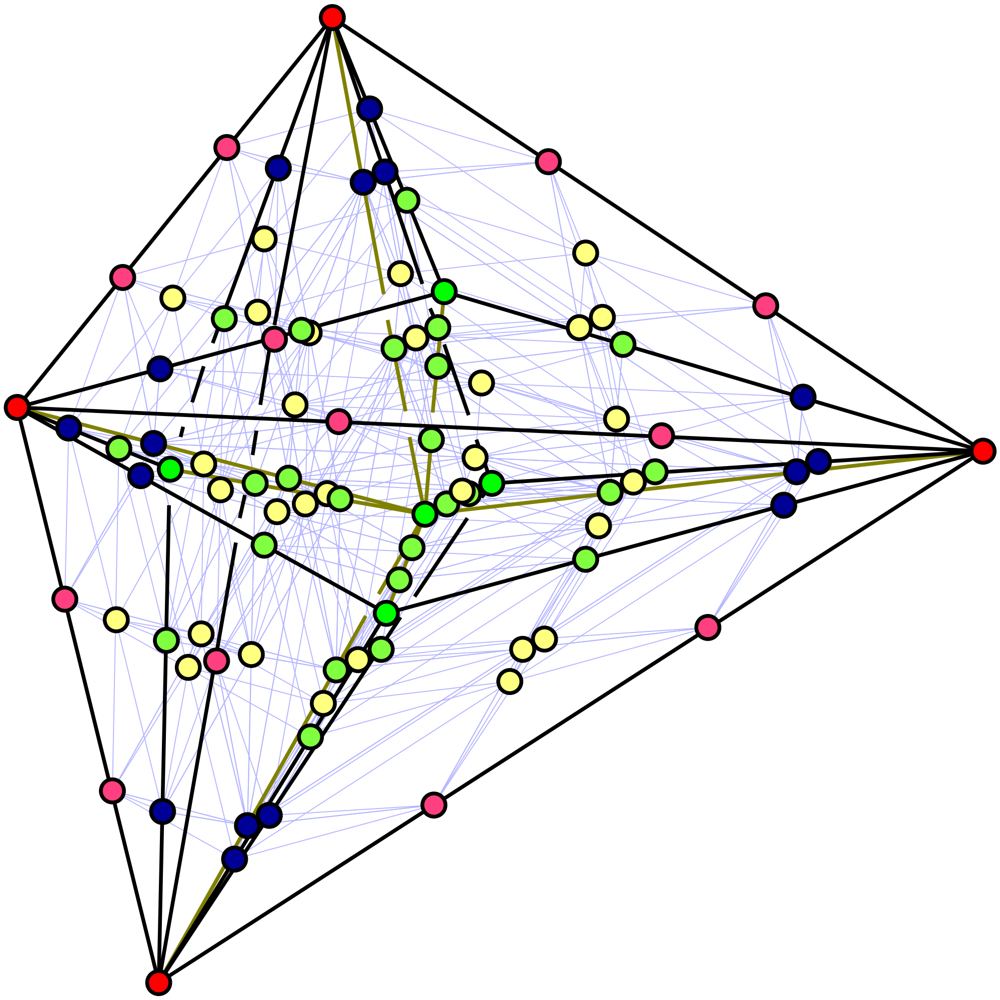

Figure 17. Worsey-Farin Scheme.

Worsey-Piper (1988)

- Apply PS-6 on faces
- Connect to interior point of tetrahedron
- Get 24 micro tetrahedra
- $r = 1$, $d = 2$
- 16 data, function and gradient at vertices.
- quadratic precision
- geometric constraints

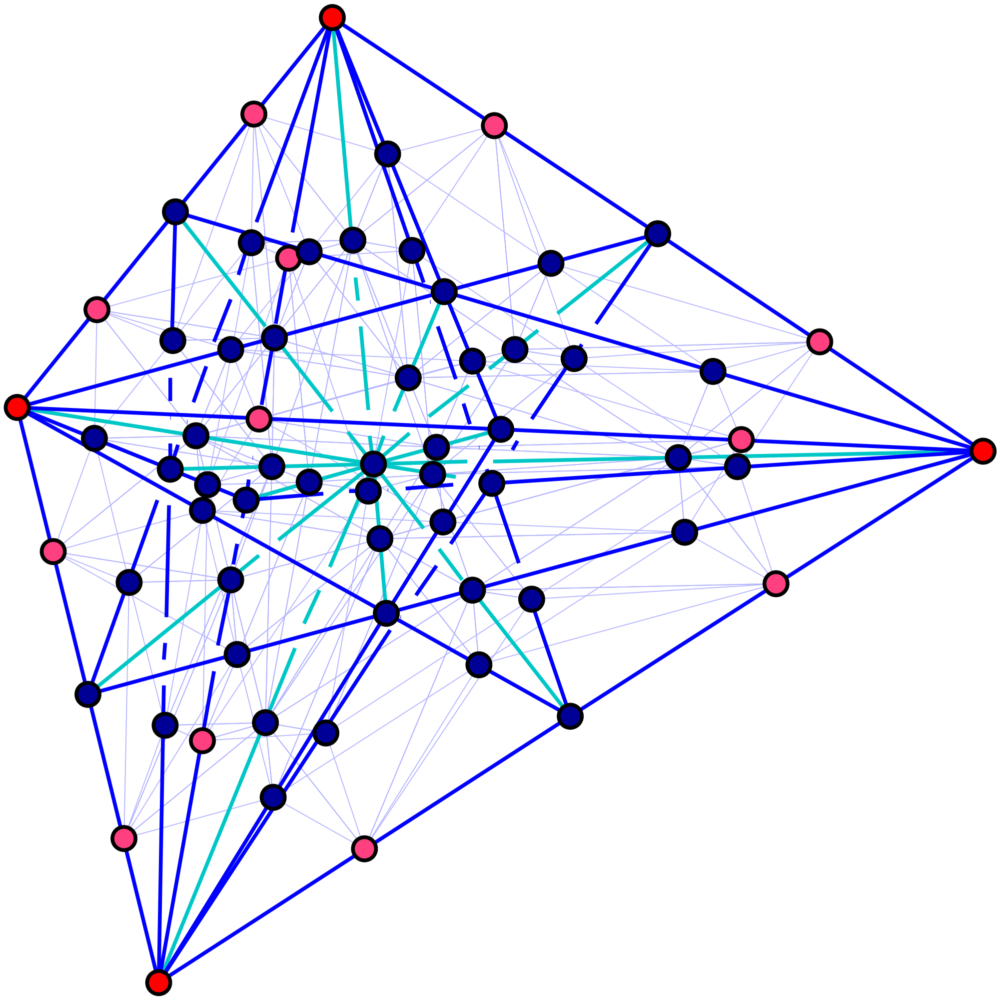

Figure 18. Worsey-Piper Scheme.

T_{60}, (Alfeld and Sorokina, 2009)

- Refine WF
- Get 60 micro tetrahedra
- $r = 1$, $d = 3$
- 28 data, function and gradient at vertices, derivative values at edges.
- cubic precision
- **no geometric constraints**

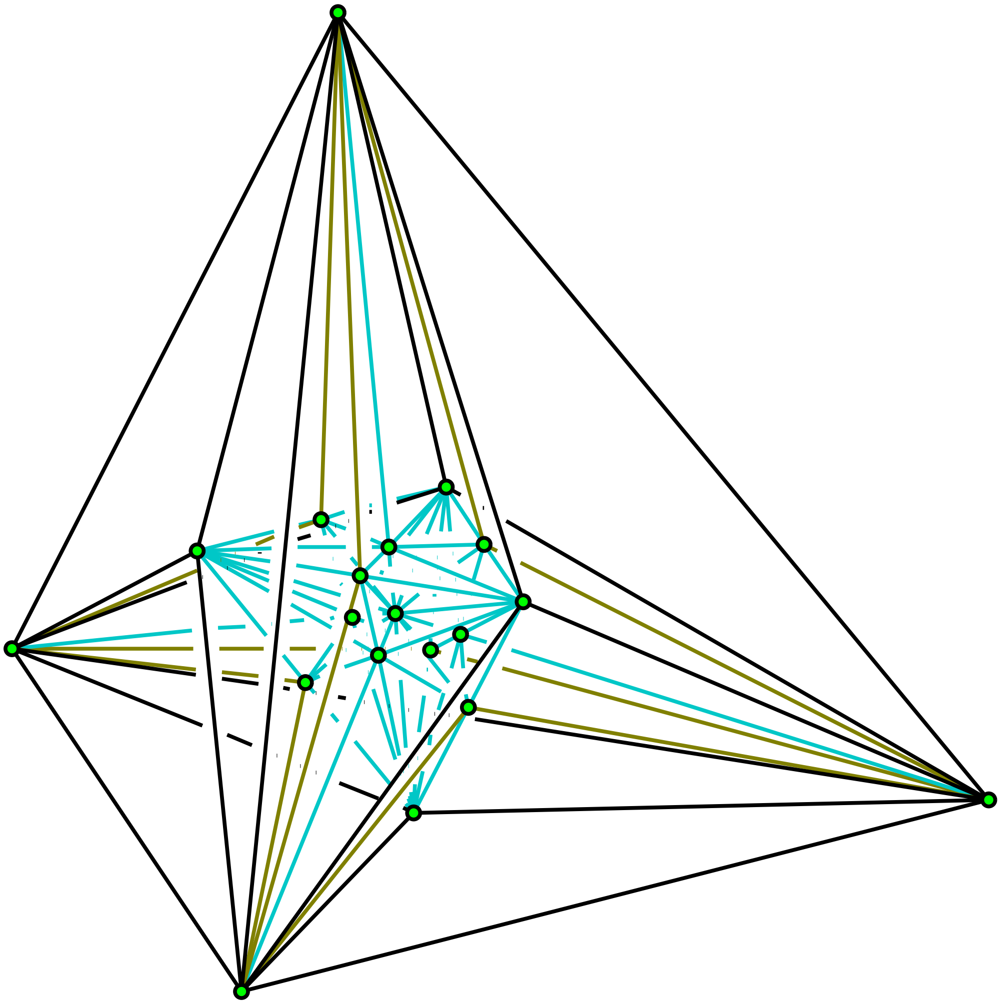

Figure 19. T_{60} Scheme.

Schumaker-Sorokina-Worsey, 2008

- Get 504 micro tetrahedra. Note!
- $r = 1$, $d = 2$
- 44 data, on Vertices, Edges, Faces. (Later condensed to 16 vertex data without loss of precision.)
- quadratic precision
- **no geometric constraints**

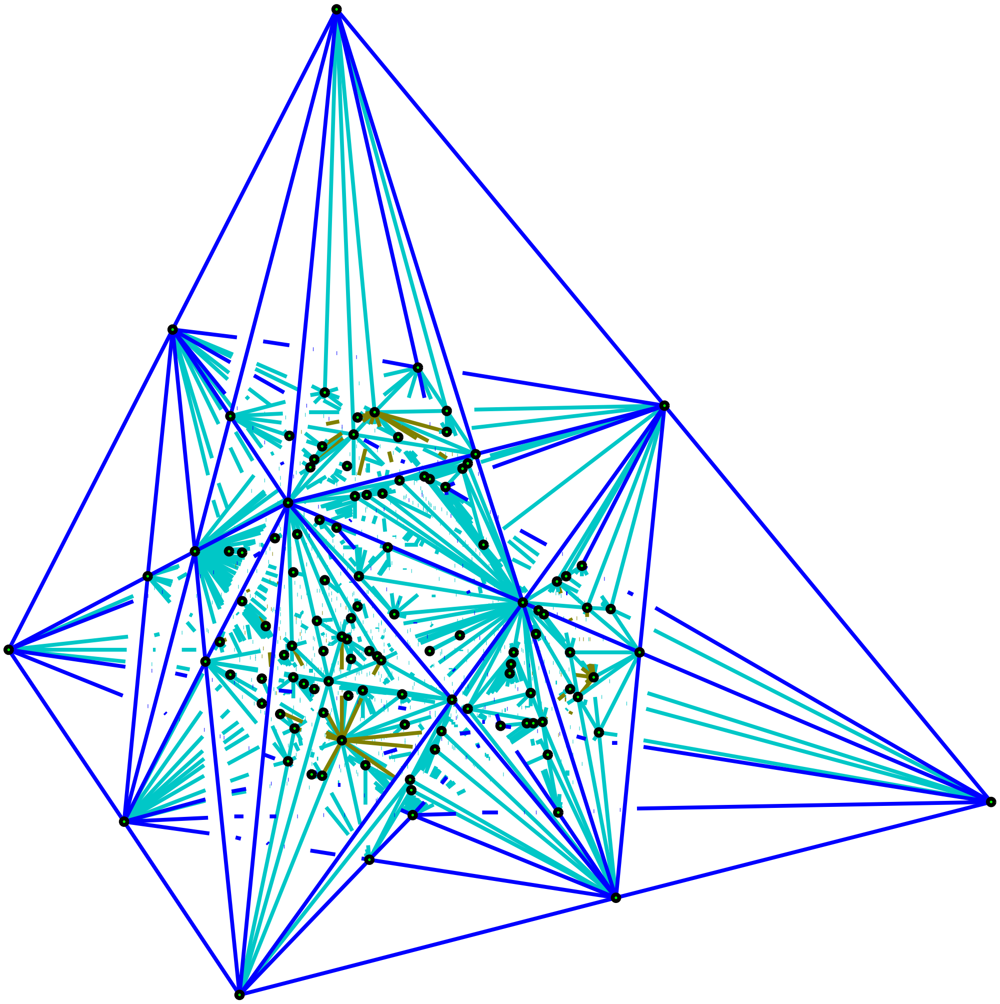

Figure 20. Schumaker-Sorokina-Worsey Scheme.

$k = 3$, $r = 1$, **Summary**

Scheme	d	N	α	data	s	constraints
Z, 1973	9	1	4	220 on F, E, V	220	no
A, 1984	5	4	2	40 on V	121	no
WF, 1987	3	12	1	28 on E, V	91	yes
WP, 1987	2	24	1	16 on V	65	yes
ScSoW, 2008	2	504	1	44 on F, E, V	761	no
ASo, 2008	3	60	1	44 on F, E, V	313	no
ASo, 2008	3	60	1	28 on E, V	313	no
AScSo, 2009	2	504	1	16 on V	761	no

A=Alfeld, F=Farin, Sc=Schumaker,

So = Sorokina, W=Worsey, Z =Ženišek.

d = polynomial degree

N = number of microtetrahedra

α = highest degree of derivative data

data: on faces (F), edges (E), and vertices (V)

s = (size) number of coefficients

constraints: on the location of the vertices or split points.

A Short Reading List

- *B-Form Basics* by Carl de Boor, 1987.
- *Spline Functions on Triangulations* by Ming-Jun Lai and Larry Schumaker, 2007.
- *Two Condensed Macro-Elements with Full Approximation Power* by Peter Alfeld, Larry Schumaker, and Tanya Sorokina, 2009.
- For an extended reading list, use the references in the above.

Software

- `http://www.math.utah.edu/~pa/MDS/`
- `http://www.math.utah.edu/~pa/3DMDS/`
- `http://www.math.utah.edu/~pa/tp/`

It's lunch time ...