On one extremal property of a regular simplex.

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dedicated to the memory of Borislav Bojanov

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Motivation.

1. In Geometry: problem of approximation of convex bodies by polytopes (of best approximation, inscribed, circumscribed, with fixed number of vertices or faces etc.)
2. In Approximation Theory: problems of approximation of functions by piecewise linear functions (interpolation, best approximation, best one-sided approximation etc.) defined over simplices

One way to construct asymptotically optimal sequence of triangulations.

1. Construct intermediate approximation of the convex body surface by piecewise quadratic function
2. Approximate the piecewise quadratic function by the piecewise linear function
3. Solve the following local optimization problem:
   Let a quadratic function $Q(\mathbf{x}) = \mathbf{x}\mathbf{x}^t$ and a simplex $\mathcal{T}$ in $\mathbb{R}^d$ of unit volume be given.
   Find a simplex $\mathcal{T}^*$, for which the error of best (one-sided, asymmetric, etc.) L_p-approximation of function Q by linear functions defined on $\mathcal{T}$ is minimal.
4. Use the optimal simplex to construct locally optimal mesh
5. Combine (glue) local meshes to obtain an element of a sequence of asymptotically optimal mesh
6. Move to the next element of the sequence

Motivation for considering the asymmetric norm.

1. Practical
   - strict constraints are sometimes too expensive to satisfy
   - asymmetric approximation allows more control
2. Theoretical
   - solution to the problem with strict constraints is non-stable and non-regular
   - uniqueness of the solution is guaranteed under less restrictive assumptions

Notation.
For $f \in L_p(G)$ and a subspace H of $L_p(G)$ denote by

$$E(f;H)_{L_p(G)} := \inf\{\|f-u\|_{L_p(G)} \ : \ u \in H\}$$

best approximation of the function f by H

$$E^{\pm}(f;H)_{L_p(G)} := \inf\{\|f-u\|_{L_p(G)} \ : \ \pm u \le \pm f, \ u \in H\}$$

best one-sided approximation of the function f by H

$$|f(\mathbf{x})|_{\alpha,\beta} := \alpha f_+(\mathbf{x}) + \beta f_-(\mathbf{x}), \text{ where } g_\pm(\mathbf{x}) := \max\{\pm g(\mathbf{x}); 0\}.$$

$\|f\|_{L_{p;\alpha,\beta}(G)} := \|\alpha f_+ + \beta f_-\|_{L_p(G)}$ asymmetric $L_{p;\alpha,\beta}$-norm

The connection. (V. Babenko'82)

$$\lim_{\beta\to+\infty} E(f;H)_{L_{p;1,\beta}(G)} = E^{+}(f;H)_{L_p(G)},$$
$$\lim_{\alpha\to+\infty} E(f;H)_{L_{p;\alpha,1}(G)} = E^{-}(f;H)_{L_p(G)}. \qquad (1)$$

Because of the relation

$$\|f-u\|_{p;1,\beta}^{p} = \|f-u\|_{p}^{p} + (\beta^{p}-1)\,\|(f-u)_{-}\|_{p}^{p}, \qquad \beta > 1,$$

the problem of the best $(1,\beta)$-approximation can be considered as the problem of the best approximation with non-strict constraint $f \le u$. This constraint is allowed to be violated, but the penalty

$$(\beta^{p}-1)\,\|(f-u)_{-}\|_{p}^{p}$$

for its violation is introduced into the error measure. In what follows we shall allow the value $+\infty$ for α or β, in that case identifying $E(f;H)_{L_{p;\alpha,\beta}(G)}$ with the corresponding one-sided approximation.

More notation.

Let

$$\mathcal{S}_1(G) := \{g(\mathbf{x}) = \mathbf{a}\mathbf{x}^t + c \ : \ \mathbf{a} \in \mathbb{R}^d, c \in \mathbb{R}, \mathbf{x} \in G\}.$$

The space $\mathcal{S}_1(G)$ will be the main approximation set.

Let also $\mathcal{T} = \{\mathbf{t}^1, \ldots, \mathbf{t}^{d+1}\}$ be the d-dimensional simplex with vertices $\mathbf{t}^j$, $j = 1, \ldots, d+1$.

Questions.

Let $Q(\mathbf{x}) = \mathbf{x}\mathbf{x}^t$, and for $\mathcal{T} \subset \mathbb{R}^d$, set

$$\sigma_{p;\alpha,\beta;d}(\mathcal{T}) := \frac{E(Q;\mathcal{S}_1(\mathcal{T}))_{L_{p;\alpha,\beta}(\mathcal{T})}}{|\mathcal{T}|^{1+\frac{1}{p}}},$$

where $|\mathcal{T}|$ stands for the d-dimensional volume of the simplex $\mathcal{T}$.

Problem 1. *Find*

$$\sigma_{p;\alpha,\beta;d} := \inf_{\mathcal{T}} \sigma_{p;\alpha,\beta;d}(\mathcal{T}), \tag{2}$$

and describe all simplices $\mathcal{T}$, for which the infimum in the right hand part of (2) is achieved.

A solution to Problem 1 will allow to solve the following related problems.

Problem 2. *Find*

$$\sigma_{p;d} := \inf_{\mathcal{T}} \frac{E(Q;\mathcal{S}_1(\mathcal{T}))_{L_p(\mathcal{T})}}{|\mathcal{T}|^{1+\frac{1}{p}}}, \tag{3}$$

and describe all simplices $\mathcal{T}$, for which the infimum in the right hand part of (4) is achieved.

Problem 3. *Find*

$$\sigma^{\pm}_{p;d} := \inf_{\mathcal{T}} \frac{E^{\pm}(Q;\mathcal{S}_1(\mathcal{T}))_{L_p(\mathcal{T})}}{|\mathcal{T}|^{1+\frac{1}{p}}}, \tag{4}$$

and describe all simplices $\mathcal{T}$, for which the infimum in the right hand part of (4) is achieved.

Note.

$$\sigma^{+}_{p;d} = \lim_{\beta\to+\infty} \sigma_{p;1,\beta;d}, \qquad \text{and} \qquad \sigma^{-}_{p;d} = \lim_{\alpha\to+\infty} \sigma_{p;\alpha,1;d}.$$

The quantity $\sigma^{+}_{p;d}$ has been considered for $d = 2$ in connection with the problem of finding the best triangulation $\triangle_N$ consisting of N triangles of the set $G \subset \mathbb{R}^2$ provided that the L_p-error of interpolation at the vertices of $\triangle_N$ of convex function f is minimized.

Known History.

Problem 3:

1. $d = 2$, $p = \infty$ (D'Azevedo and Simpson, 1989);
2. $d \geq 2$, $p = \infty$ (Rajan, 1991);
3. $d = 3$, $p = 2$ (Brezin, 1992);
4. $d = 2$, $p = 1$ (Böröczky, Ludwig, 1999);
5. $d = 2$, $p = 2$ (Pottmann et al, 2000);
6. $d \geq 2$, $p \in \mathbb{N}$ (Chen, 2007);
7. $d = 2$, $p \in (0, \infty)$ (V. Babenko, Yu. Babenko, and Skorokhodov, 2008);
8. $d \geq 2$, $p \in (1, \infty)$ (Chen, 2008).

Problem 2:

1. $d = 2$, $p = 2$ and $\alpha = \beta = 1$ (Nadler, 1986)

Problem 1:

1. all $\alpha, \beta > 0$, $1 \leq p \leq \infty$ and $d \in \mathbb{N}$ (YB, V. Babenko, N. Parfinovych, D. Skorokhodov, 2009)

Main Result

(solution to Problem 1 for $\alpha, \beta > 0$, $d \in \mathbb{N}$ and $1 \leq p \leq \infty$)

Theorem. *Let $\mathcal{T}_0$ be a regular simplex of unit volume in $\mathbb{R}^d$. Let $\alpha, \beta > 0$, $d \in \mathbb{N}$ and $1 \leq p \leq \infty$. Then*

$$\sigma_{p;\alpha,\beta;d} = \sigma_{p;\alpha,\beta;d}(\mathcal{T}_0).$$

Steps of the proof.

The proof of the main theorem consists of two parts:

Lemma 1. *Let $\mathcal{T}$ be an arbitrary d-dimensional simplex of unit volume. Then there exists a constant $C > 0$, independent of $\mathcal{T}$, such that*

$$\sigma_{p;\alpha,\beta;d}(\mathcal{T}) \geq C(\operatorname{diam} \mathcal{T})^2.$$

Lemma 2. *If $\mathcal{T}$, $\mathcal{T} \neq \mathcal{T}_0$, is a simplex of unit volume in $\mathbb{R}^d$ then there exists a simplex $\mathcal{T}^* \subset \mathbb{R}^d$ of unit volume such that*

$$\sigma_{p;\alpha,\beta;d}(\mathcal{T}) > \sigma_{p;\alpha,\beta;d}(\mathcal{T}^*).$$

Indeed, in view of Lemma 1, there exists an optimal d-dimensional simplex $\mathcal{T}'$ of unit volume such that $\sigma_{p;\alpha,\beta;d} = \sigma_{p;\alpha,\beta;d}(\mathcal{T}')$. Then, Lemma 2 gives $\mathcal{T}' = \mathcal{T}_0$.

Applications.

Corollary 1. *Let* $d \in \mathbb{N}$ *and* $1 \leq p \leq \infty$. *Then*

$$\sigma_{p;d} = \frac{E(Q;\mathcal{S}_1(\mathcal{T}_0))_{L_p(\mathcal{T}_0)}}{|\mathcal{T}_0|^{1+\frac{1}{p}}}.$$

Corollary 2. *Let* $d \in \mathbb{N}$ *and* $1 \leq p \leq \infty$. *Then*

$$\sigma^{\pm}_{p;d} = \frac{E^{\pm}(Q;\mathcal{S}_1(\mathcal{T}_0))_{L_p(\mathcal{T}_0)}}{|\mathcal{T}_0|^{1+\frac{1}{p}}}.$$

Recall that $E^{+}(Q;\mathcal{S}_1(\mathcal{T}_0))$ is the error of linear interpolation of function Q.