Smooth planar r-splines of degree $2r$

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Δ= connected finite simplicial complex supported on $|\Delta| \subset \mathbb{R}^2$. $r \geq 0$ integer.

Space of splines: the $\mathbb{R}-$ vector space $C_k^r(\Delta)$: $\{F : |\Delta| \longrightarrow \mathbb{R} : F|_\sigma =$ poly. of deg $\leq k, \forall \sigma \in \Delta_2$, and $F \in C^r\}$.

Goal: Find $\dim_{\mathbb{R}} C_k^r(\Delta)$.

Billera-Rose [Disc. Comput. Geom.,1991]: use commutative and homological algebra: consider $\mathbb{R}^2$ embedded in $\mathbb{R}^3$, let $\widehat{\Delta}$= cone of Δ with origin in $\mathbb{R}^3$ and $R = \mathbb{R}[x, y, z]$.

$C^r(\widehat{\Delta}) = \{F : |\widehat{\Delta}| \longrightarrow \mathbb{R} : F|_{\widehat{\sigma}} \in R, \forall \sigma \in \Delta_2$, and $F \in C^r\}$.

$C^r(\widehat{\Delta})$ is a finitely generated graded $R-$module and $\dim_{\mathbb{R}} C_k^r(\Delta) = \dim_{\mathbb{R}} C^r(\widehat{\Delta})_k$= the dimension of the degree k piece of $C^r(\widehat{\Delta})$.

$R = \mathbb{R}[x,y,z] = \bigoplus_{d\in\mathbb{Z}} R_d$, where R_d= homog. polys. of degree d. $R = \cdots 0 \oplus \mathbb{R} \oplus R_1 \oplus \cdots$.

M is a graded $R-$module if $M = \bigoplus_{d\in\mathbb{Z}} M_d$ s.t. if $p \in R_d$ and $m \in M_e$, then $pm \in M_{d+e}$.

Examples: (1) $R(-i)$ is a graded $R-$module (we think of the generator 1 as being in degree i instead of 0). $R(-i)_d = R_{d-i}$. (2) $I \subset R$ homogeneous ideal (i.e., generated by homogeneous elements) with the same grading as R. Similarly, R/I.

M is f.g. graded $R-$module, then we have the **Hilbert function** $HF(M,d) = \dim_{\mathbb{R}} M_d$ and the **Hilbert series** $HS(M,t) = \sum HF(M,d)t^d$. For example: $HF(R,d) = \binom{2+d}{d}$ and $HS(R(-i),t) = \frac{t^i}{(1-t)^3}$.

If $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ is a graded exact sequence of $R-$modules (the maps preserve the grading), then $HF(N,d) = HF(M,d) + HF(P,d)$.

Graded free resolutions: M a graded $R-$module minimally generated by $m_1, \ldots, m_n$ of degrees $d_1, \ldots, d_n$. How far is M from being a free $R-$module?

$0 \longrightarrow \ker\phi \longrightarrow R^n \xrightarrow{\phi} M \longrightarrow 0$ is an exact sequence of $R-$modules, where $\phi(e_i) = m_i$, and $\{e_i\}=$ the canonical basis.

$\ker\phi = \{\sum a_i e_i \in R^n | \sum a_i m_i = 0\}=$ the *First Syzygies module* is an $R-$submodule of R^n, minimally generated by p elements. Repeat the process. In the end we get an exact complex of free $R-$modules

$$0 \longrightarrow \cdots \longrightarrow R^p \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

We want to preserve the grading, so $\deg(e_i) = d_i$ and so forth, to get (with Hilbert Syzygy Theorem)

$$0 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with $F_j \simeq \bigoplus R(-a_{jl})$ (e.g., $F_0 \simeq \bigoplus_1^n R(-d_i)$).

Billera-Rose [Disc. Comput. Geom.,1991] We have a (graded) exact sequence:

$0 \to C^r(\hat{\Delta}) \to R^t \oplus R^{e_0}(-r-1) \to R^{e_0} \to M \to 0$

where $t =$ number of triangles, $e_0 =$ number of interior edges, and M is the cokernel of the middle map.

Schenck-Stillman [J. Pure Appl. Alg.,1997]

$$0 \to N \to M \to \bigoplus_{v \in \Delta_0^0} R/J(v) \to 0$$

where: $\Delta_0^0 =$ the interior vertices, $J(v) = \langle l_1^{r+1}, \ldots, l_{n_v}^{r+1} \rangle$, $l_1, \ldots, l_{n_v}$ are the linear forms vanishing on $\hat{\epsilon}$, $\epsilon =$ the edges of Δ having v as a vertex, and N is an $R-$module of finite length (i.e., for $k >> 0$, $HF(N,k) = 0$).

Schenck-Stillman [Adv. Appl. Math.,1997]: $HF(R/J(v),k)$ is computed so

$$\dim C_k^r(\Delta) = L(\Delta, r, k) + HF(N,k)$$

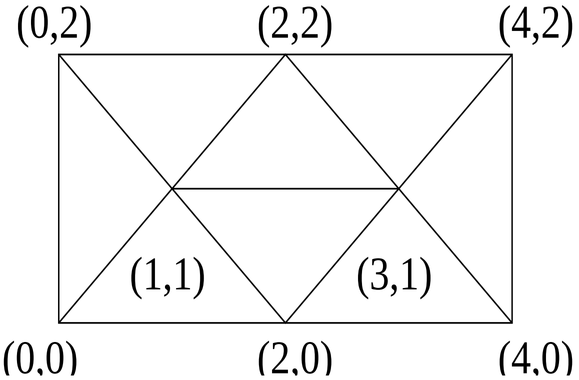
$L(\Delta, r, k) =$ the *Alfeld-Schumaker formula* (it involves both combinatorics and geometry-slopes of the linear forms). If $k >> 0$, $\dim C_k^r(\Delta) = L(\Delta, r, k)$.

How big k can be?

Alfeld-Schumaker [Numer. Math.,1990]: $k \geq 3r+1$, for any strongly connected planar simplicial complex Δ.

Schenck-Stiller [Manuscr. Math.,2002] conjectured that formula holds for $k \geq 2r+1$.

S.T. [J. Approx. Theory,2005] showed that if



then for any r, $HF(N, 2r) \neq 0$.

Ko-Stiller showed that for some particular values for r and same triangulation, the conjecture holds and they checked the conjecture for other types of triangulations.

$I_i = \mathcal{J}(v_i)$ for v_1 and v_2 the interior vertices.

$$I_1 = \langle (x+y-2z)^{r+1}, (x-y)^{r+1}, (y-z)^{r+1} \rangle$$

$$I_2 = \langle (x+y-4z)^{r+1}, (x-y-2z)^{r+1}, (y-z)^{r+1} \rangle$$

With a change of coordinates we get

$$I_1 = \langle x^{r+1}, (x+y)^{r+1}, y^{r+1} \rangle$$

$$I_2 = \langle z^{r+1}, (z+y)^{r+1}, y^{r+1} \rangle$$

The minimal free resolutions are:

$$0 \to R^2 \xrightarrow{\left[\begin{smallmatrix} A_1 & D_1 \\ B_1 & E_1 \\ C_1 & F_1 \end{smallmatrix}\right]} R^3 \to R \to R/I_1 \to 0$$

and

$$0 \to R^2 \xrightarrow{\left[\begin{smallmatrix} A_2 & D_2 \\ B_2 & E_2 \\ C_2 & F_2 \end{smallmatrix}\right]} R^3 \to R \to R/I_2 \to 0.$$

By Schenck-Stillman [J. Pure Appl. Alg.,1997], $N \simeq R(-r-1)/\langle C_1, F_1, C_2, F_2 \rangle$, so we need $HF(R/\langle C_1, F_1, C_2, F_2 \rangle, r-1) \neq 0$.

Proof for $r+1=2n$

Key observations: Think of I_1 as ideal in $A=\mathbb{R}[x,y]$ and I_2 as ideal in $A'=\mathbb{R}[y,z]$, so $C_1, F_1 \in A$ and $C_2, F_2 \in A'$. Replacing x by z in C_1 and F_1 we obtain C_2 and F_2.

Graded free resolution for I_1:

$$0 \to A(-3n)^2 \overset{\left[\begin{smallmatrix} A_1 & D_1 \\ B_1 & E_1 \\ C_1 & F_1 \end{smallmatrix}\right]}{\longrightarrow} A(-2n)^3 \to I_1 \to 0$$

so $\deg C_1 = \deg F_1 = 3n-2n = n$.

$\langle C_1, F_1\rangle$ is a complete intersection so $HS(A/\langle C_1, F_1\rangle, t) = \frac{1-2t^n+t^{2n}}{(1-t)^2} = (1+t+\cdots+t^{n-1})^2$. So $HF(A/\langle C_1, F_1\rangle, 2n-2) = 1 \neq 0$.

Let $x^u y^v, u+v = 2n-2 = r-1$ not in $\langle C_1, F_1\rangle$. Suppose $x^u y^v \in \langle C_1, F_1, C_2, F_2\rangle$.

$$x^u y^v = \alpha_1 C_1 + \beta_1 F_1 + \alpha_2 C_2 + \beta_2 F_2, \alpha_i, \beta_j \in R$$

Make $z=x$, so from observations, $x^u y^v \in \langle C_1, F_1\rangle$. Contradiction.

Thank you!